Obtaining representations for continuous optimization problems (based on a recent efficient split criterion)

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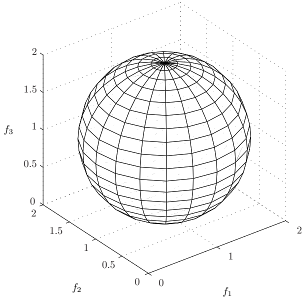
Short problem description

- Given: Continuous optimization problem

$$\min_{x \in X} f(x) = [f_1(x), \dots, f_p(x)]^\top$$

with feasible set $X \neq \emptyset, X \subseteq \mathbb{R}^n$ compact, $f : X \to \mathbb{R}^p$ continuous (no convexity assumption!)

- Sought: Nondominated set $Z_N \subseteq f(X)$ (Pareto front)



$\Rightarrow$ Since Z_N contains infinitely many points:

- Generate **representation** of Z_N (finite set of nondominated points)
- by using a scalarization technique

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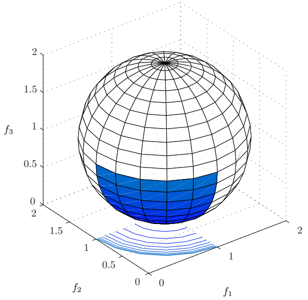

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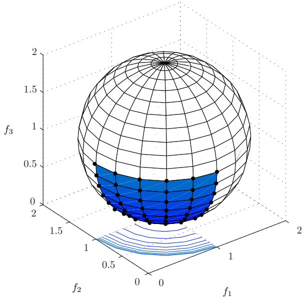

⇒ Since Z_N contains infinitely many points:

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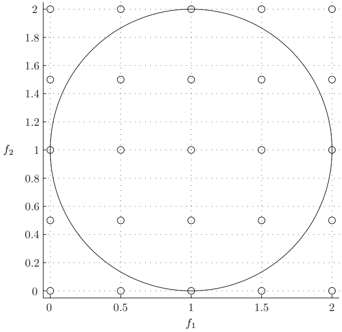
A-priori approach with equidistant parameters

- build equidistant parameter grid
- solve one scalarization for each grid point
- Example:

ε-constraint method :

$$\begin{array}{ll} \min & f_3(x) \\ \text{s.t.} & f_1(x) \leq \varepsilon_1 \\ & f_2(x) \leq \varepsilon_2 \\ & x \in X \end{array}$$

with $\varepsilon \in \mathbb{R}^2$



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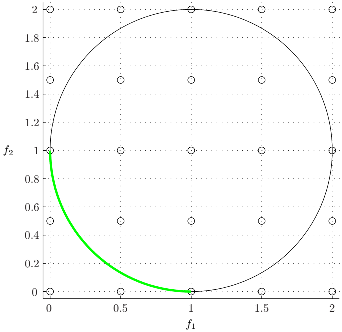

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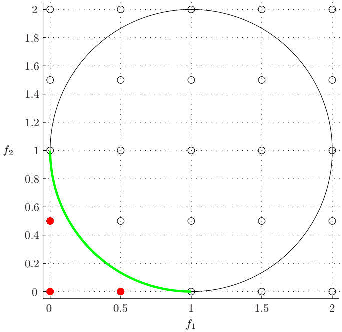

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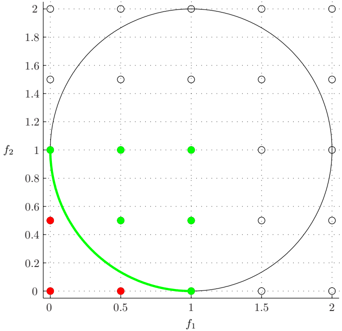

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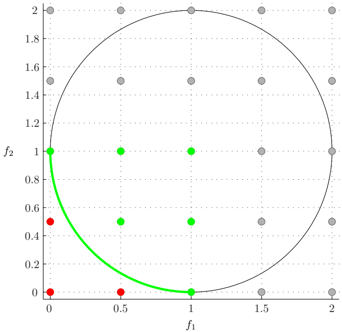

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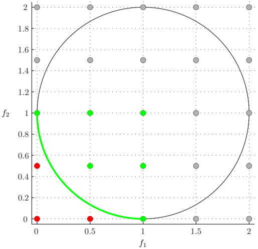

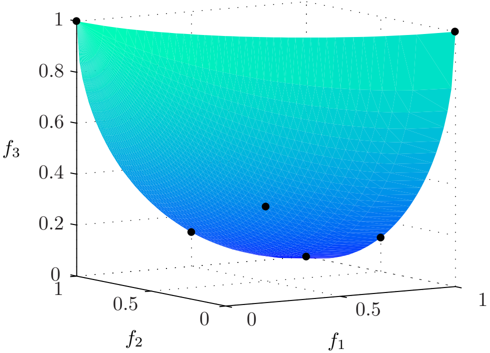

⇒ only 6 of 25 subproblems yield nondominated points

A-priori parametric algorithm

Advantages:

- easy implementation

Disadvantages:

- infeasibility: no contribution to the representation
- redundancy: no new information obtained
- fixed number of subproblems

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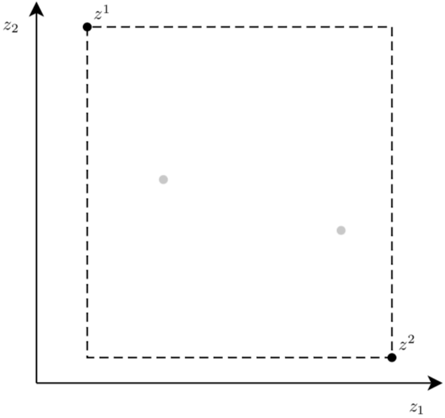
Adaptive approaches

Idea of adaptive approaches:

- take representing points that have already been computed into account
- construct parameters of next subproblem dependent on them

Decomposition into disjoint boxes

Bicriteria case:



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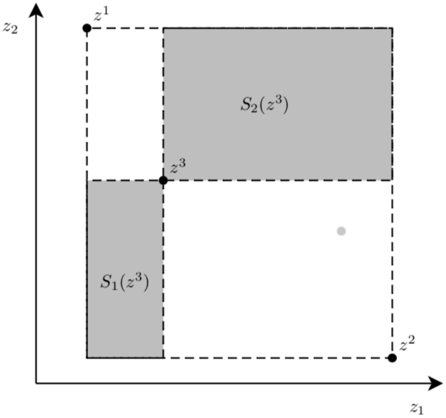

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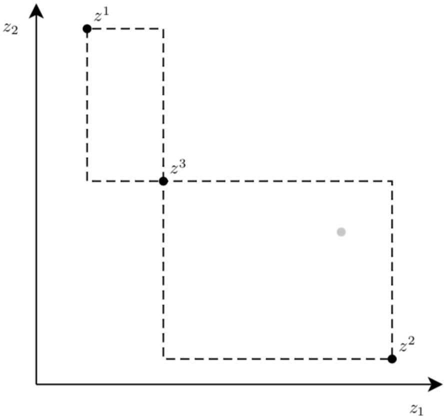
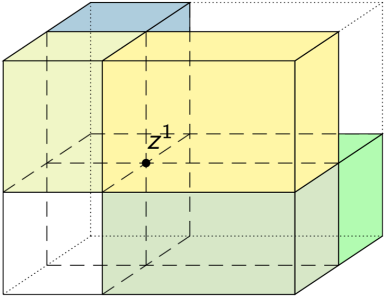


Decomposition into disjoint boxes

Multicriteria case:



Algorithmic variants

LV:

- adaptive
- decomposition into disjoint boxes
- weighted Tchebycheff with local ideal point as reference point and weights according to shape of box
- select box with largest volume
- **L**ocal reference point (lower corner of box), **V**olume-based selection

EC:

- a-priori
- ε-constraint method
- equidistant grid: 5×5, 7×7, 10×10

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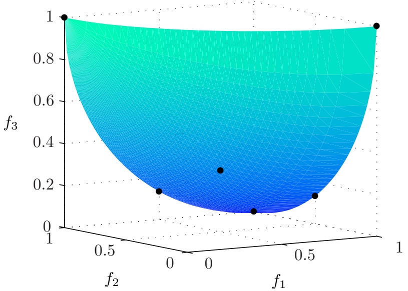
Test case 1: sphere problem



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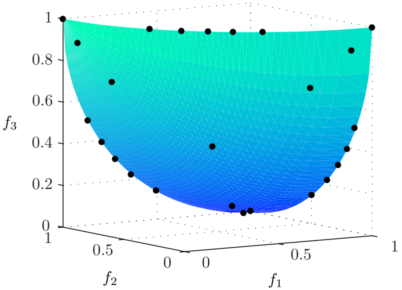

(f) LV

# SP		EC	LV
25 (+3)	card	6	25
	rdh	66.43%	80.48%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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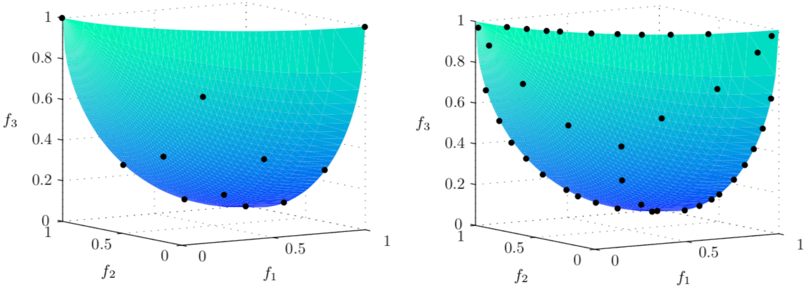

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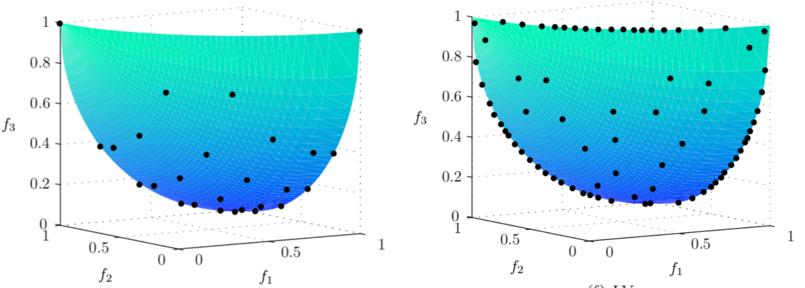

(a) EC

(f) LV

# SP		EC	LV
100 (+3)	card	26	76
	rdh	77.75%	83.66%

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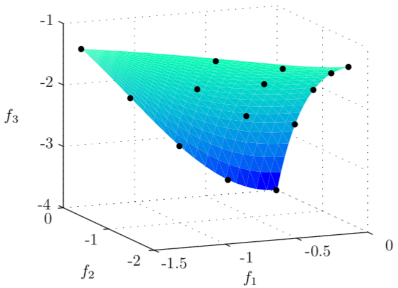
Test case 2



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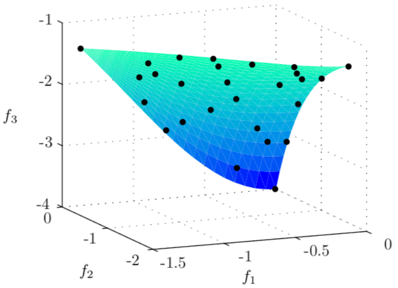

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# SP		EC	LV
25 (+3)	card	14	27
	rdh	6.43%	9.62%

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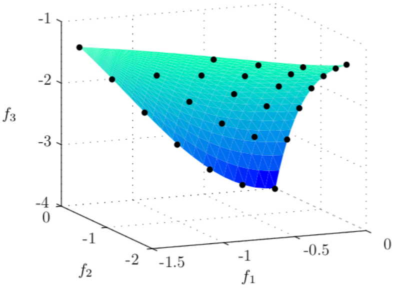

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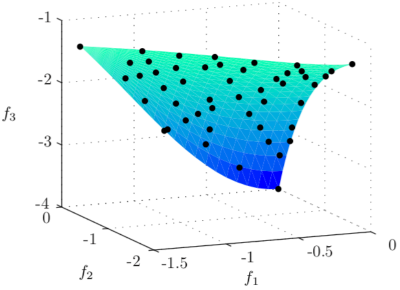

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49 (+3)	card	26	49
	rdh	9.10%	11.25%

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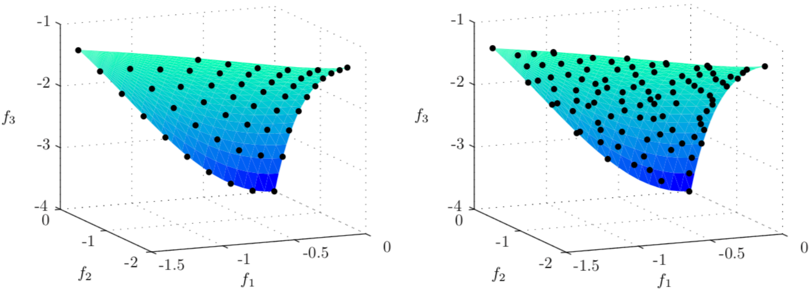

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100 (+3)	card	52	100
	rdh	11.14%	12.56%

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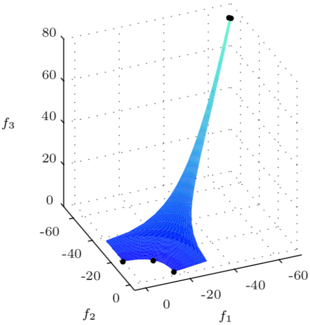
Test case 3: Comet problem



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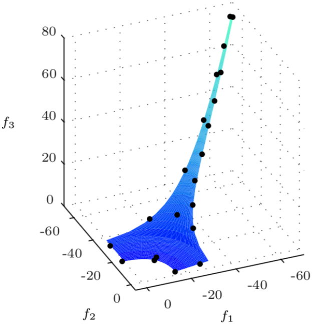

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# SP		EC	LV
25 (+4)	card	5	21
	rdh	71.08%	87.18%

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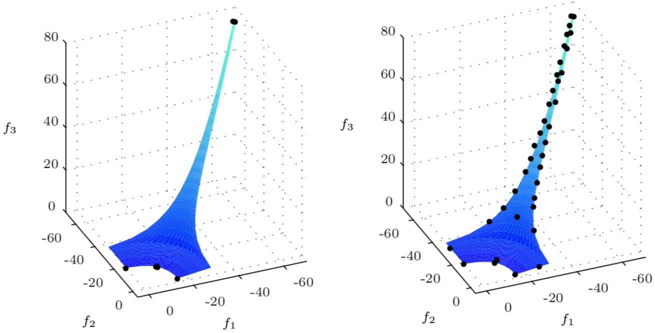

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# SP		EC	LV
49 (+4)	card	6	37
	rdh	71.09%	87.96%

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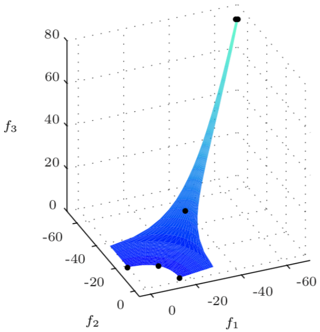

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100 (+4)	card	6	73
	rdh	77.81%	88.44%

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Experience

Advantages:

- rather easy implementation
- high ratio of points contributing to the representation

Disadvantages:

- number of boxes 'exploses' (e.g. sphere problem, 100 subproblems: 1715 boxes at termination)

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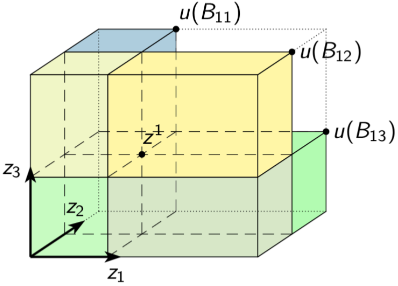
New decomposition

- recent advances in the discrete case (complete representation)
- decomposition of search region into smallest possible set of (non-disjoint) rectangular sets
- lower corner fixed to global ideal point
- avoid redundancy by use of a particular split criterion



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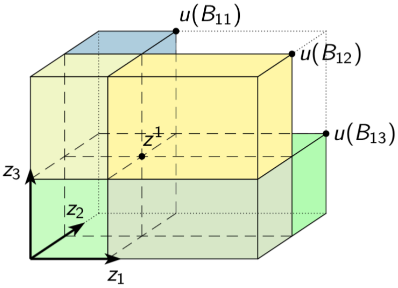

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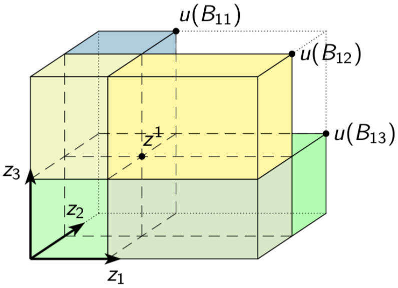

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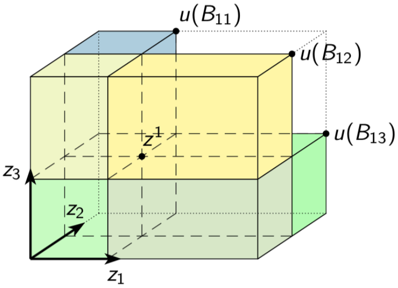

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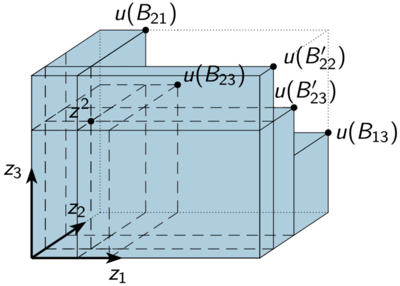


Complexity

- general case: $\mathcal{O}(N^{\lfloor p/2 \rfloor})$ for $p \geq 2$.
- tricriteria case : Explicit bound on number of boxes after insertion of N points: $2N + 1$

D., Klamroth (2015)
A linear bound on the number of scalarizations needed to solve discrete tricriteria optimization problems
JOGO 61:643–676

Klamroth, Lacour, Vanderpooten (2015)
On the representation of the search region in multi-objective optimization
EJOR 245(3):767–778

D., Klamroth, Lacour, Vanderpooten (2017)
Efficient computation of the search region in multi-objective optimization
EJOR 260(3):841–855

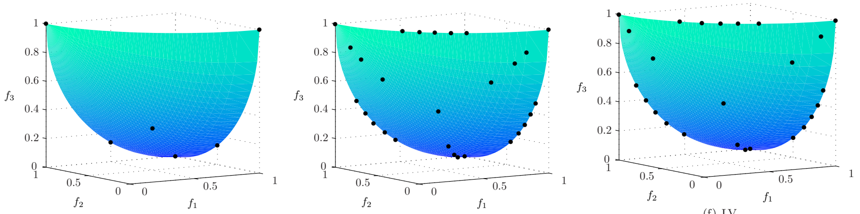
Algorithmic variants

volume-based: Select box with largest volume (corrected by volume of empty space of nondominated points at boundary)

GV:

- adaptive
- weighted Tchebycheff with global ideal point as reference point
- **G**lobal reference point, **V**olume-based selection

Test case 1: Sphere problem



(a) EC (b) GV (f) LV

# SP		EC	GV	LV
25 (+3)	card	6	28	25
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SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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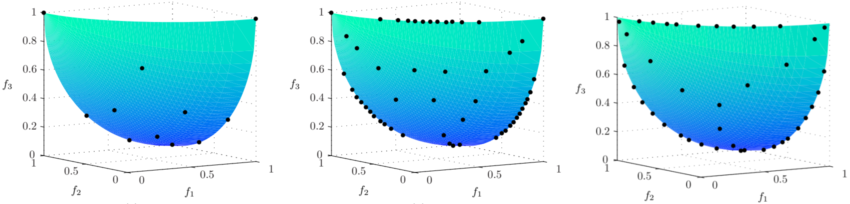

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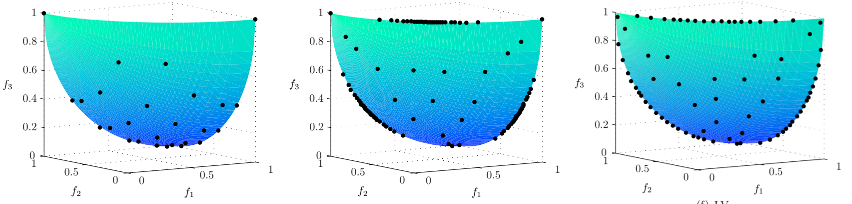

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# SP		EC	GV	LV
100 (+3)	card	26	103	76
	rdh	77.75%	83.05%	83.66%

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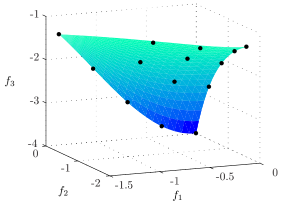
Test case 2



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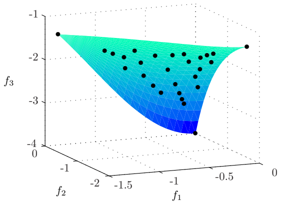

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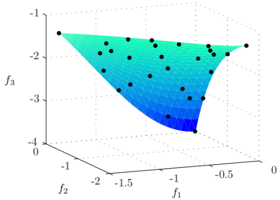

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25 (+3)	card	14	28	27
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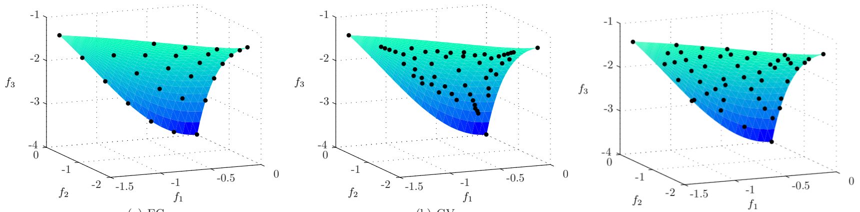

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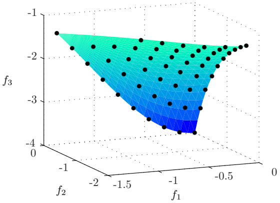

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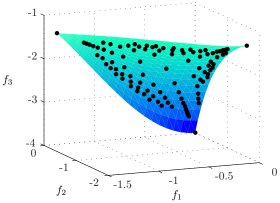

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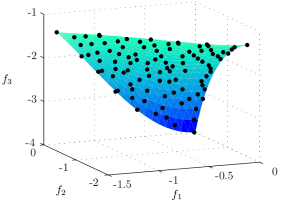

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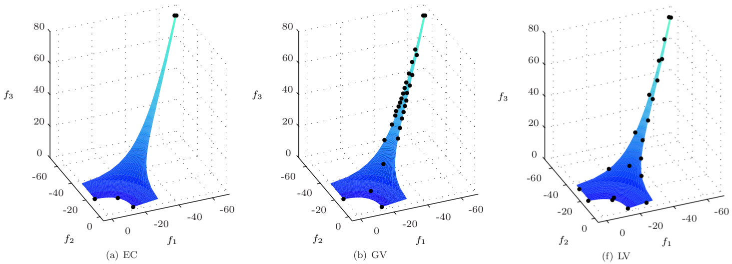

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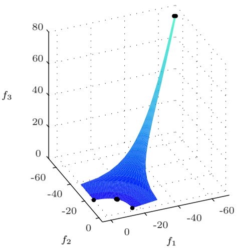
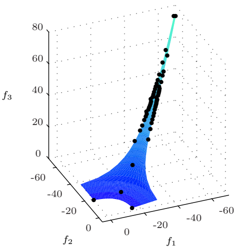
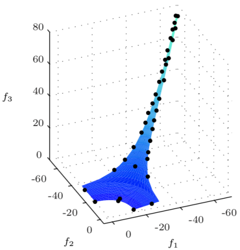


(a) EC



(b) GV

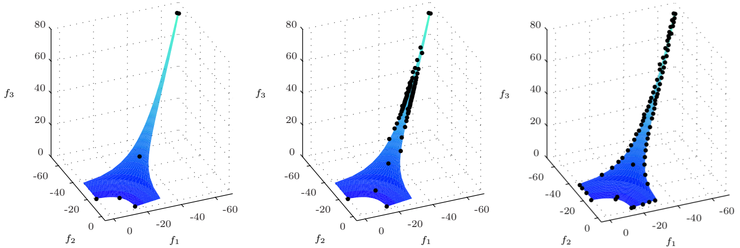


(f) LV

# SP		EC	GV	LV
49 (+4)	card	6	53	37
	rdh	71.09%	87.01%	87.96%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

Test case 3: Comet problem



(a) EC (b) GV (f) LV

# SP		EC	GV	LV
100 (+4)	card	6	78	73
	rdh	77.81%	87.15%	88.44%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

Experience

Advantages:

- high ratio of points contributing to the representation
- fewer boxes to be handled compared to disjoint boxes

Disadvantage:

- tends to clustering

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Algorithmic variants

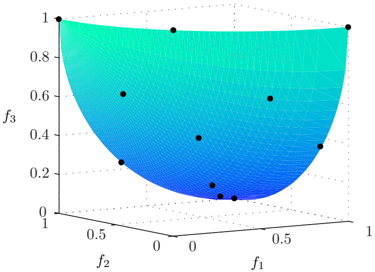
hypervolume-based approach: for each box solve scalarization, compute contribution to dominated hypervolume of obtained point, select point with largest contribution
(note: more points are computed than inserted)

GH

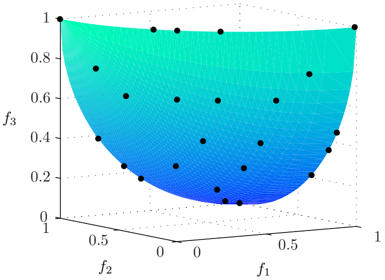
- adaptive
- weighted Tchebycheff
- **G**lobal reference point, **H**ypervolume-based selection

GH2: same as GH, but insert all computed points at the end

Test case 1: Sphere problem



(c) GH



(d) GH2

# SP		EC	GH	GH2	LV
25 (+3)	card	6	11	24	25
	rdh	66.43%	77.02%	80.77%	80.48%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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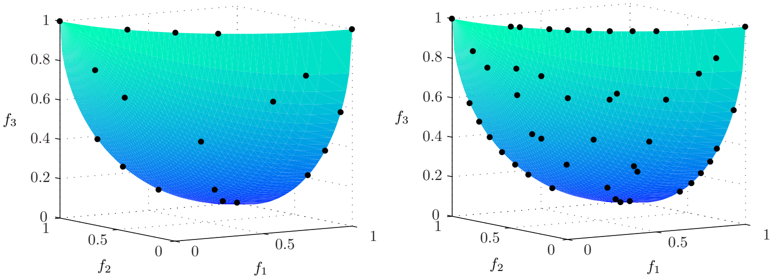

(c) GH (d) GH2

# SP		EC	GH	GH2	LV
49 (+3)	card	11	19	45	40
	rdh	73.39%	80.06%	82.67%	82.28%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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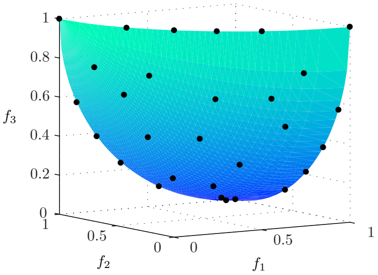
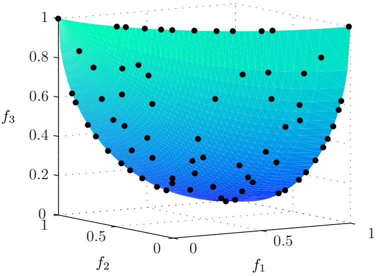


(c) GH

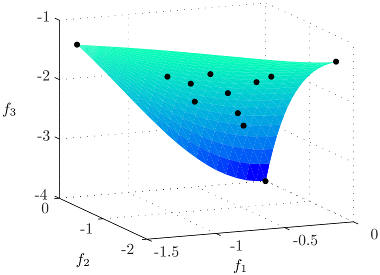


(d) GH2

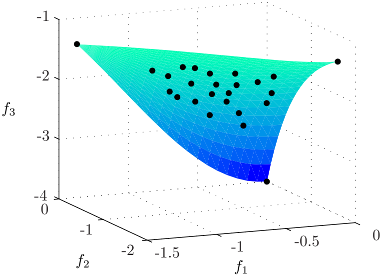
# SP		EC	GH	GH2	LV
100 (+3)	card	26	29	71	76
	rdh	77.75%	81.69%	83.42%	83.66%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

Test case 2



(c) GH



(d) GH2

# SP		EC	GH	GH2	LV
25 (+3)	card	14	12	25	27
	rdh	6.43%	8.05%	9.52%	9.62%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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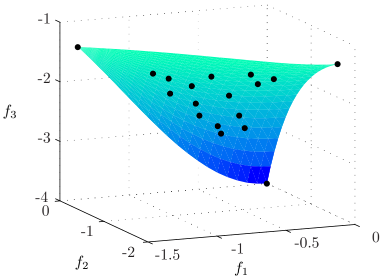
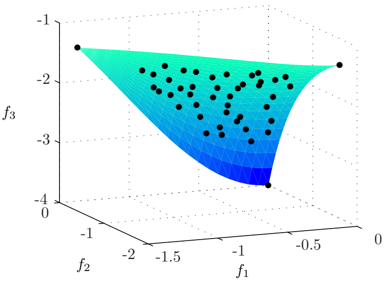


(c) GH



(d) GH2

# SP		EC	GH	GH2	LV
49 (+3)	card	26	18	45	49
	rdh	9.10%	9.21%	10.89%	11.25%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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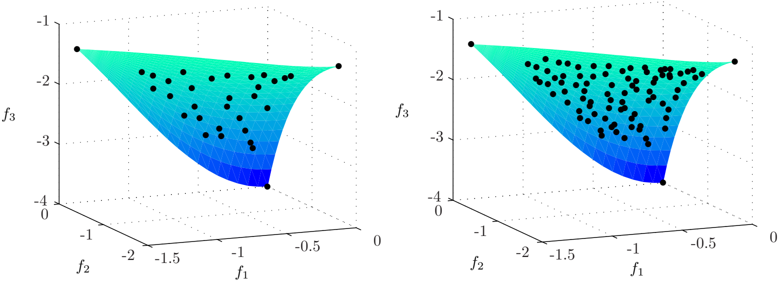
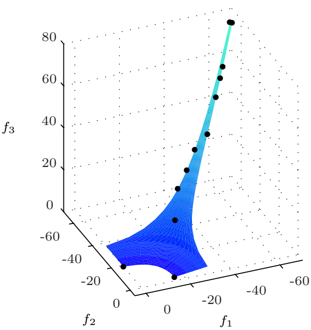


(c) GH (d) GH2

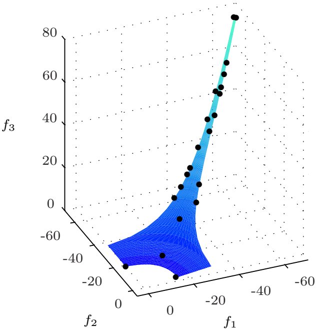
# SP		EC	GH	GH2	LV
100 (+3)	card	52	32	87	100
	rdh	11.14%	10.56%	11.97%	12.56%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

Test case 3: Comet problem



(c) GH



(d) GH2

# SP		EC	GH	GH2	LV
25 (+4)	card	5	12	21	21
	rdh	71.08%	86.21%	86.82%	87.18%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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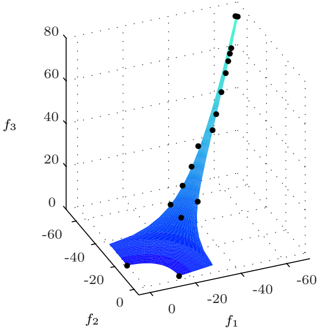

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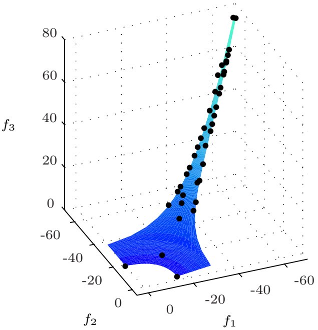

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# SP		EC	GH	GH2	LV
49 (+4)	card	6	17	39	37
	rdh	71.09%	86.83%	87.42%	87.96%

SP: subproblems, card: cardinality, rdh: relative dominated hypervolume

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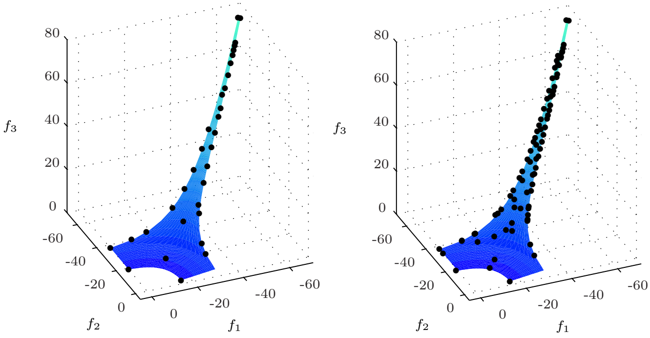

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Summary:

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- less boxes to handle than with decomposition into disjoint boxes
- results promising but to be improved
  - volume-based selection produces clusters
  - hypervolume-based selection performs quite well but overhead of solved subproblems

Extensions and future work:

- additional use of local lower bounds
- replace global upper bound after certain iterations by (estimate on) nadir point
- ...

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