Equidistant representations: connecting coverage and uniformity in biobjective optimization

Martin Kidd, Richard Lusby & Jesper Larsen

Division of Management Science, Technical University of Denmark (DTU)

Introduction

Our problem

- We have a MIP with two objectives
- A conversion to a single objective MIP can be solved in a reasonably short time
- We would like to present some (nondominated) solutions to the decision makers

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Solution approach

- Find entire nondominated set?
  - Too time consuming
  - Only a couple of solutions would suffice anyway
- Find a discrete representation of the nondominated set

Introduction

Agenda

Discrete representations:

- Some theoretical results for general biobjective problems
- A new scalarization/criterion space search method (when a single-objective black-box solver is available)

Introduction

Notation

Biobjective optimization problem:

$$\min f_1(x), f_2(x)$$

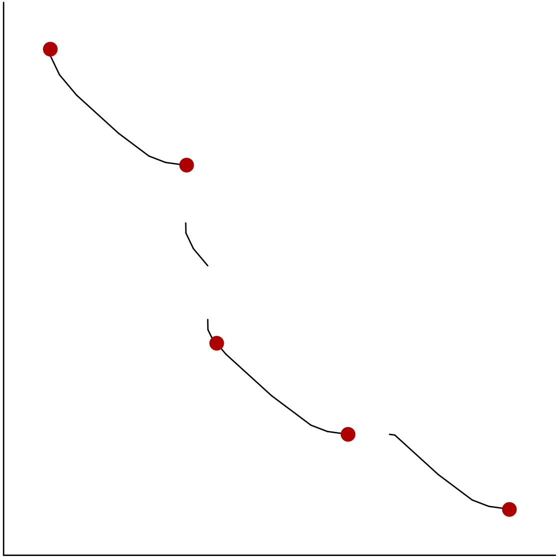
$$\text{s.t.} \qquad x \in X$$

Other notation:

Y	The set of feasible objective function values
N	The nondominated set
y	A point in Y
R	A *representation*, i.e. a finite subset of N

Introduction

Example of a representation



The discrete representation problem

Measuring the quality of a representation

The discrete representation problem

Measuring the quality of a representation

- $d : N \mapsto \mathbb{R}$ is a given distance metric
- Coverage:

$$\Gamma(R) = \max_{y \in N} \min_{y' \in R} d(y, y'),$$

- Uniformity:

$$\Delta(R) = \min_{y, y' \in R, y \neq y'} d(y, y'),$$

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The discrete representation problem (DRP)

$$\begin{aligned} \min \quad & \Gamma(R) \\ \max \quad & \Delta(R) \\ \min \quad & |R| \\ \text{s.t.} \quad & R \subset N, \quad |R| < \infty. \end{aligned}$$

Equidistant representations

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Let

$$R = \{y^1, y^2, \ldots, y^{|R|}\} \subset N$$

such that $y_1^j < y_1^{j+1}$.

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A representation R is **equidistant** if

$$d(y^j, y^{j+1}) = d(y^{j'}, y^{j'+1}) = d_R$$

for all $1 \leq j, j' \leq |R| - 1$.

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A representation is **complete** if it contains the lexicographical minima.

Equidistant representations

Critical points and critical distances

Critical points of a representation R

$$\bar{y}^j = \underset{y' \in N : y_1^j \leq y_1' \leq y_1^{j+1}}{\operatorname{argmax}} \min_{y \in R} d(y, y')$$

Equidistant representations

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Note that the coverage error can be restated in terms of critical distances as

$$\Gamma(R) = \max_{j \in \{0, \ldots, |R|\}} \bar{d}^j.$$

Equidistant representations

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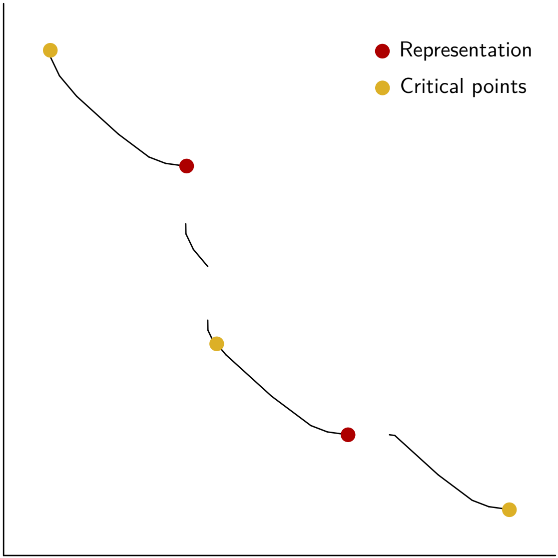
Later in this talk...

A method for finding the critical points of a representation (note they are not part of the representation itself)

Equidistant representations

Critical points



Equidistant representations

Super- and sub-representations

The union of R and its critical points is its *super-representation*:

$$\sup(R) = \{\bar{y}^0, y^1, \bar{y}^1, \dots, y^{|R|}, \bar{y}^{|R|}\}$$

Equidistant representations

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If $|R|$ is odd, the *sub-representation* of R is

$$\operatorname{sub}(R) = \{y^2, y^4, \ldots, y^{|R|-2}\},$$

Note that if R is a complete equidistant representation, then $\sup(\operatorname{sub}(R)) = R$.

Equidistant representations

Lemma

A representation R for which $\sup(R)$ is a complete equidistant representation has the minimum coverage error over all representations of cardinality $|R|$.

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Theorem

A representation R is a nondominated solution to the DRP if $\sup(R)$ is a complete equidistant representation.

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Theorem

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Corollary

If $\sup(R)$ is a complete equidistant representation

$$\Gamma(R) = d_{\sup(R)} = \Delta(\sup(R))$$

Equidistant representations

Theorem

For any representation R of the nondominated set a biobjective optimization problem,

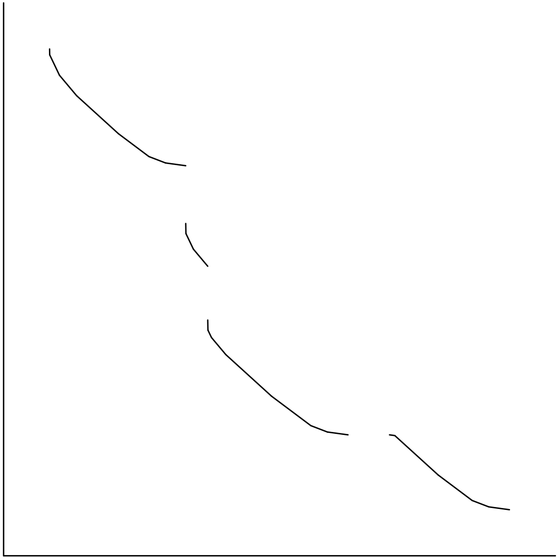
$$\Gamma(R) \geq \Delta(\sup(R))$$

and

$$\Delta(R) \leq \Gamma(\text{sub}(R))$$

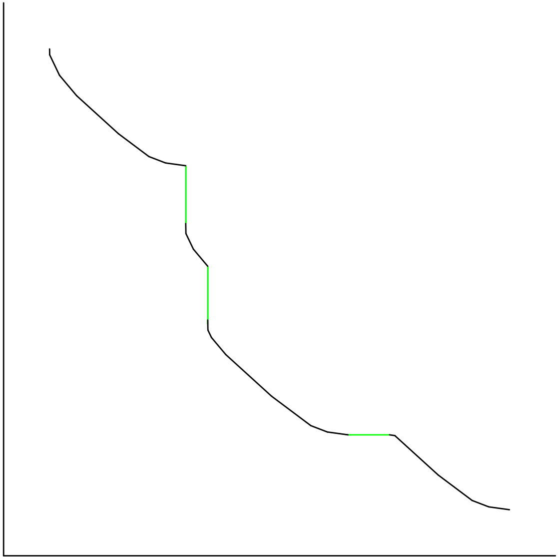
Equidistant representations

Connected relaxation



Equidistant representations

Connected relaxation



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- $\tilde{N}$: Connected relaxation of N
- For any connected nondominated set a complete equidistant representation exists
- Let $\tilde{R}$ be a complete equidistant representation of $\tilde{N}$ such that $|\tilde{R}| = |R|$.
- Since $N \subseteq \tilde{N}$, it follows from previous Lemmas, that

$$\Gamma(R, N) \geq \Gamma(\tilde{R}, \tilde{N}) = d_{\sup(\tilde{R},\tilde{N})} = \Delta(\sup(\tilde{R}, \tilde{N})) \geq \Delta(\sup(R, N)).$$

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- Similarly,

$$\Delta(R) \leq \Delta(\tilde{R}) = d_{\tilde{R}} = \Gamma(\text{sub}(\tilde{R}, \tilde{N})) \leq \Gamma(\text{sub}(R)).$$

Equidistant representations

Main implication of these results

Given a representation R, no further information on N and a black-box solver:

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Given a representation R, no further information on N and a black-box solver:

- $\sup(R)$ (and thus the critical distances $\bar{d}^j$) can be computed by solving $O(|R|)$ single-objective problems (upcoming)

Equidistant representations

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Given a representation R, no further information on N and a black-box solver:

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- An optimality gap for coverage can be obtained:

$$\frac{\Gamma(R) - \Delta(\sup(R))}{\Gamma(R)}$$

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- An optimality gap for uniformity can be obtained:

$$\frac{\Gamma(\mathrm{sub}(R)) - \Delta(R)}{\Delta(R)}$$

The Voronoi cut method

Computing super-representations

Computing the critical point inbetween two consecutive points in the representation:

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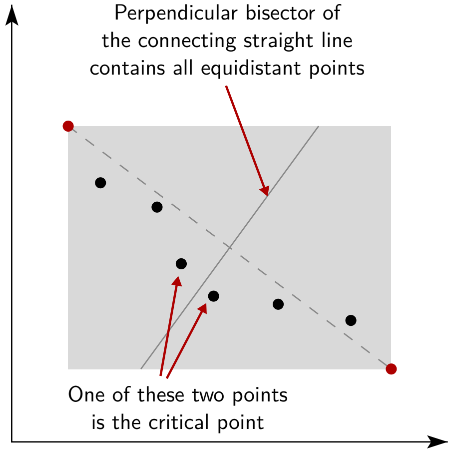
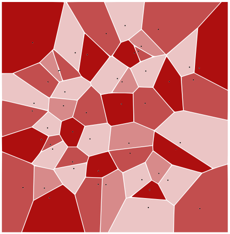
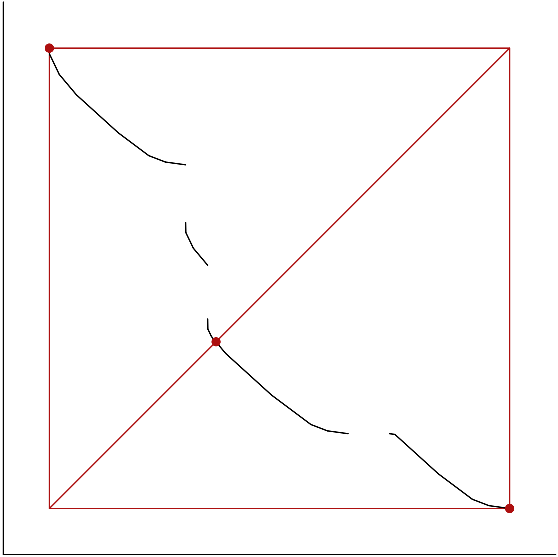


Same idea behind the construction of Voronoi diagrams:



The Voronoi cut method

Voronoi cut method for finding representations



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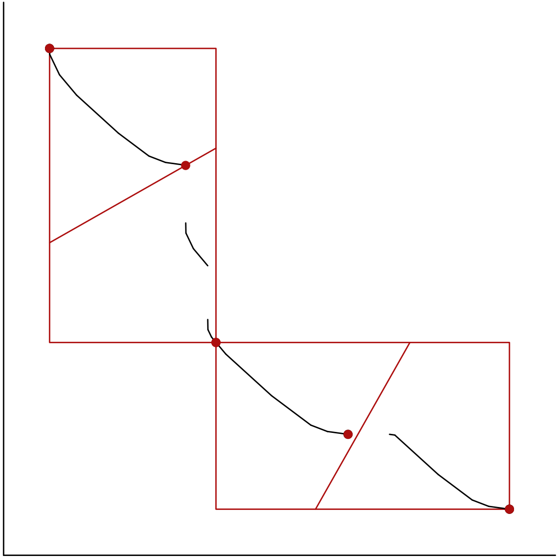
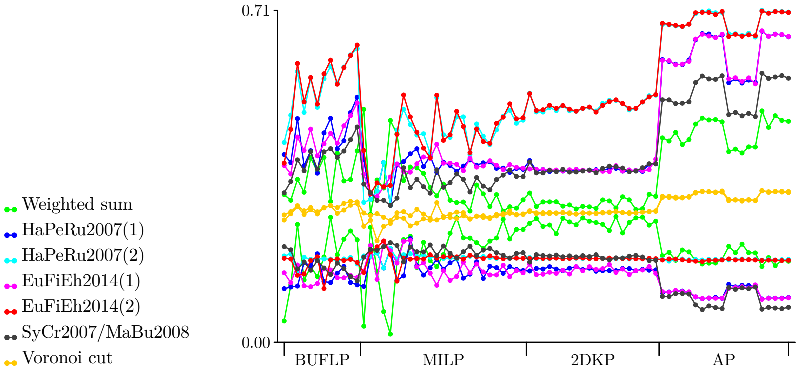


The Voronoi cut method

Comparison with existing methods

Generating a representation R consisting of 5 points:



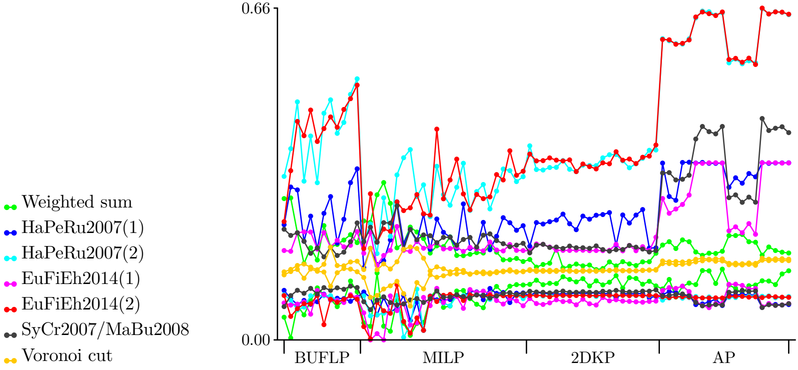
For each method,

- upper line plots the coverage error of the sub-representation
- lower line plots the uniformity

The Voronoi cut method

Comparison with existing methods

Generating a representation R consisting of 9 points:



For each method,

- upper line plots the coverage error of $\text{sub}(R)$
- lower line plots the uniformity of R

The Voronoi cut method

Computation times

		Avg time per CPLEX call			
		BUFLP	MILP	2DKP	AP
	Weighted sum	0.06	0.05	0.94	1.28
	HaPeRu2007(1)	0.20	0.27	1.25	2.43
	HaPeRu2007(2)	0.19	0.24	1.18	2.18
$n = 5$	SyCr2007	0.23	0.26	1.96	59.20
	MaBu2008	0.27	0.25	1.96	66.59
	EuFiEh2014(1)	0.19	0.20	1.20	2.43
	EuFiEh2014(2)	0.15	0.19	1.19	2.17
	Voronoi Cut	0.40	0.19	1.71	4.02
	Weighted sum	0.05	0.04	0.88	1.19
	HaPeRu2007(1)	0.22	0.27	1.16	2.75
	HaPeRu2007(2)	0.22	0.25	1.12	2.29
$n = 9$	SyCr2007	0.31	0.34	2.51	124.93
	MaBu2008	0.34	0.32	2.70	88.93
	EuFiEh2014(1)	0.23	0.24	1.12	2.79
	EuFiEh2014(2)	0.19	0.21	1.13	2.24
	Voronoi Cut	2.25	0.23	1.67	4.28

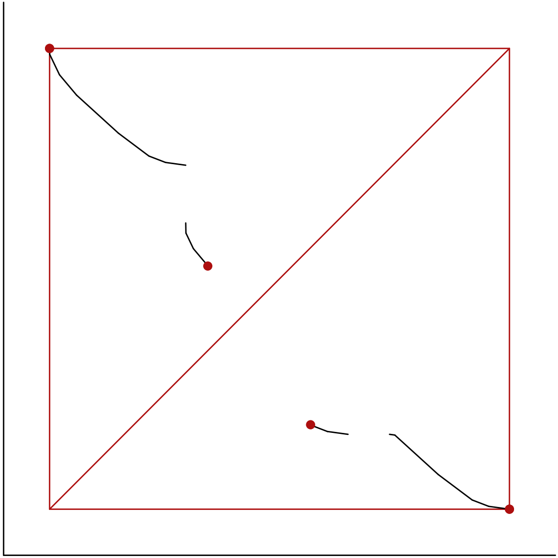
The Voronoi cut method

Solver calls

	Worst case # of calls for Voronoi cut method			Max # of calls							
				$n = 5$				$n = 9$			
		$n = 5$	$n = 9$	BUFLP	MILP	2DKP	AP	BUFLP	MILP	2DKP	AP
$p = 1$	$4 + 6(n - 2)$	22	46	18	20	22	22	40	42	44	44
$p = 2$	$4 + 6(n - 2)$	22	46	20	20	22	22	40	44	44	44
$p = \infty$	$4 + 10(n - 2)$	34	74	22	20	24	24	44	46	46	48
			Weighted sum	7				11			
			Other methods	10				18			

The Voronoi cut method

Some final thoughts on gaps in the nondominated set



Conclusion

Summary

- We present a dual relationship between coverage and uniformity that allows for the calculation of optimality gaps
- We present a method for finding critical points of a representation, which can be used to calculate coverage in absence of the full nondominated set
- This method can also be used to generate representations, and outperforms existing scalarization/criterion space search methods

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Looking forward

Can similar theoretical results be found for more than two objecties?

Conclusion

Questions?