Quantifying the hardness of the enumeration of Pareto optima: a theoretical framework with application to scheduling problems

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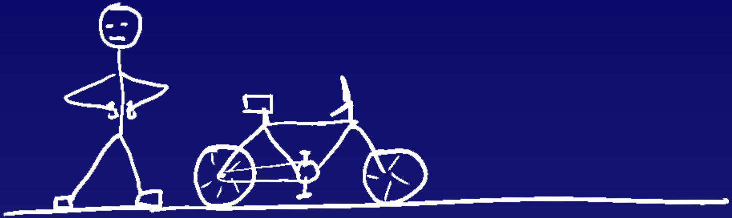
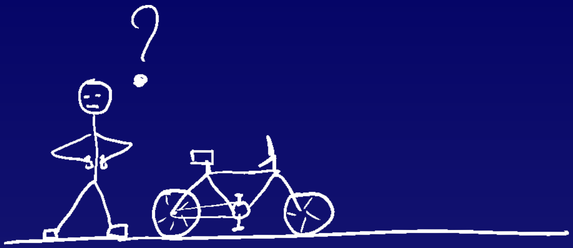
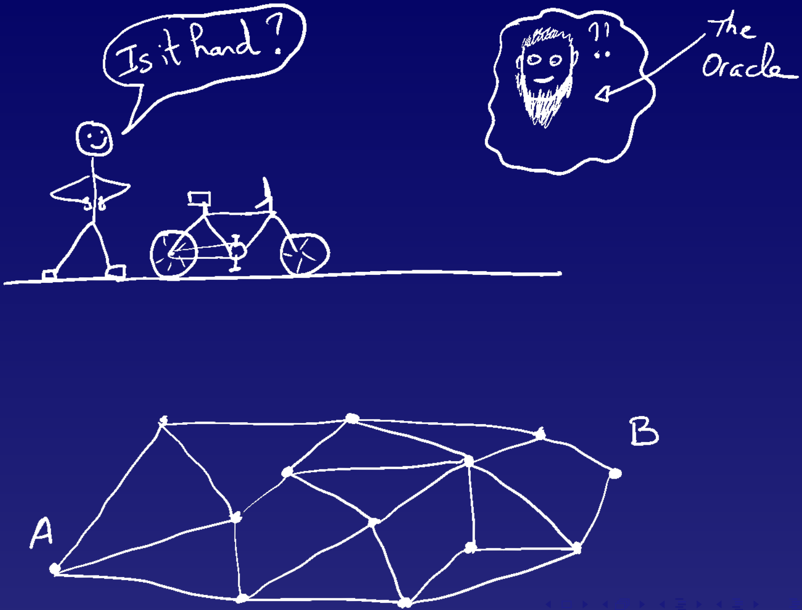
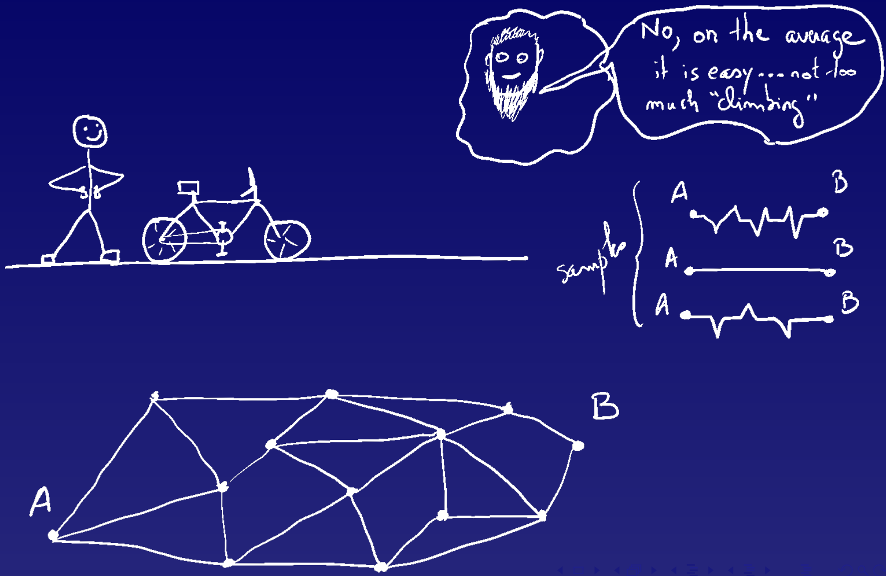
Michele GARRAFA (Doctor, Politecnico di Torino, It).



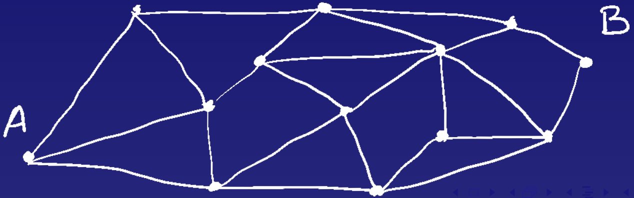


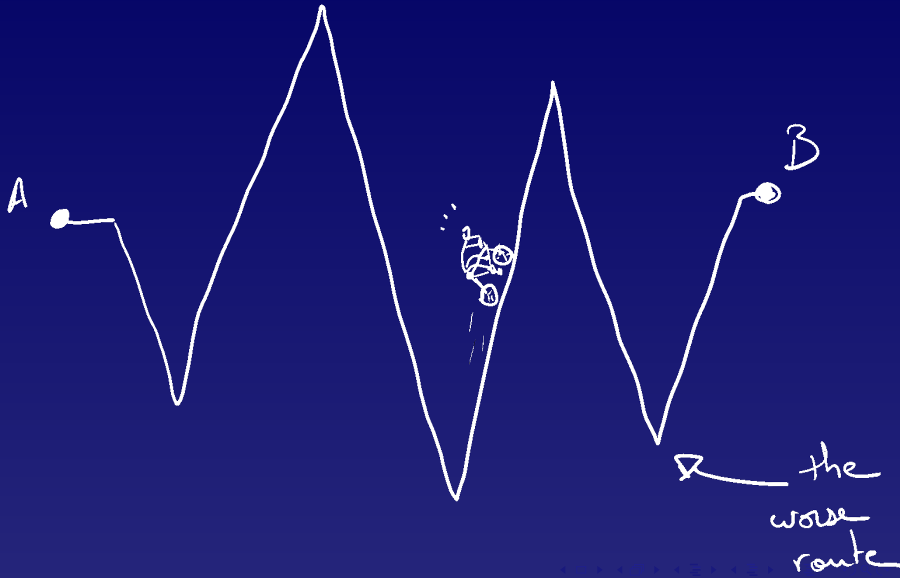

A
B

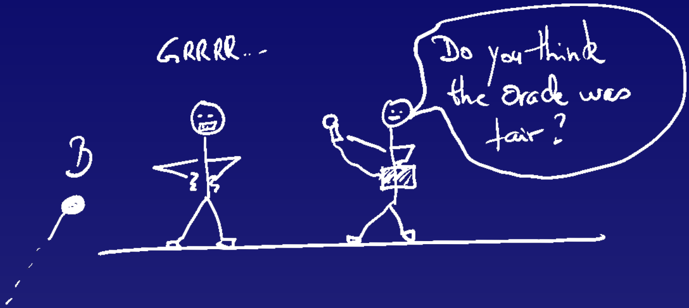

Is it hard ?
The Oracle
?!
A
B


No, on the average
it is easy... not too
much "climbing"
samples
A
B
A
B
A
B
B
A


Great! I go at random!


A
B


A
B
the
worse
route


GRRRR...
Do you think the Oracle was fair?

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How can we quantify this complexity ?

$\Rightarrow$ *Exponential-time algorithms* are tools to answer this question.

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- What is called an "exponential algorithm"?....
- For a NP-hard problem, an exact algorithm for which *the worst-case (time/space) complexity can be computed*.
- Find "theoretical" algorithms with worst-case time/space upper bounds as low as possible...

The MIS problem has been shown to be solvable $O^*(2^n)$ in 1977, $O^*(1.381^n)$ in 1999, $O^*(1.2201^n)$ in 2009, ...

NB : $O^*(exp(n)) = O(poly(n)exp(n))$

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- Why designing exponential time/space algorithms (ETA) ?
  - The beauty of the game,
  - To provide a quantitative information on the difficulty of a NP-hard problem,
  - Because, in a short future, ETA will start to beat in practice heuristics !

    $O^*(1.2201^n)$ is faster than $O(n^4)$ for $n \leq 90$,

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  - 3-SAT : $O^*(1.3211^n)$ time (Iwama et al., 2010),
  - Hamiltonian circuit : $O^*(1.657^n)$ time (Bjorklund, 2010),
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- What about scheduling problems (single machine) ?

Problem	brute force	wctc	wcsc	Reference
$1\|dec\|f_{max}$	$O^*(n!)$	$O^*(2^n)$	exp	[1]
$1\|dec\|\sum_i f_i$	$O^*(n!)$	$O^*(2^n)$	exp	[1]
$1\|prec\|\sum_i C_i$	$O^*(n!)$	$O^*((2-\epsilon)^n)$	exp	[2]
$1\|prec\|\sum_i w_i C_i$	$O^*(n!)$	$O^*(2^n)$	exp	[3]
$1\|d_i\|\sum_i w_i U_i$	$O^*(n!)$	$O^*(2^n)$ $O^*(1.4142^n)$	exp exp	[3] [4]
$1\|d_i\|\sum_i T_i$	$O^*(n!)$	$O^*(2^n)$	exp	[3] & [4]
$1\|d_i\|\sum_i w_i T_i$	$O^*(n!)$	$O^*(2^n)$	poly	[5]
$1\|r_i, prec\|\sum_i w_i C_i$	$O^*(n!)$	$O^*(3^n)$	exp	[3] & [4]

[1] F. Fomin, D. Kratsch (2010). Exact Exponential Algorithms, Springer.

[2] M. Cygan, M. Philipczuk, M. Philipczuk, J. Wojtaszczyk (2011). Scheduling partially ordered jobs faster than 2^n, ESA 2011.

[3] G. Woeginger (2003). Exact algorithms for NP-hard problems : A survey, in M. Junger, G. Reinelt, G. Rinaldi (Eds) : Combinatorial Optimization – Eureka I shrink !, Springer, LNCS 2570.

[4] C. Lenté, M. Liedloff, A. Soukhal, V.T'kindt (2013). On an extension of the Sort & Search method with application to scheduling theory, Theoretical Computer Science, 511, pp 13-22.

[5] M. Garraffa, L . Shang, F. Della Croce, V.T'kindt (2018). An exact exponential branch-and-merge algorithm for the single machine total tardiness problem, Theoretical Computer Science, 745, pp 133-149.

- What about scheduling problems (others) ?

Problem	brute force	wctc	wcsc	Reference
$P\|dec\|f_{max}$	$O^*(m^n n!)$	$O^*(3^n)$	exp	[4]
$P\|dec\|\sum_i f_i$	$O^*(m^n n!)$	$O^*(3^n)$	exp	[4]
$P4\|\|C_{max}$	$O^*(4^n)$	$O^*(2.4142^n)$	exp	[4]
$P3\|\|C_{max}$	$O^*(3^n)$	$O^*(1.7321^n)$	exp	[4]
$P2\|\|C_{max}$	$O^*(2^n)$	$O^*(1.4142^n)$	exp	[4]
$P2\|d_i\|\sum_i w_i U_i$	$O^*(3^n)$	$O^*(1.7321^n)$	exp	[4]
$F2\|\|C^k_{max}$	$O^*(2^n)$	$O^*(1.4142^n)$	exp	[4]
$F3\|\|C_{max}$	$O^*(n!)$	$O^*(3^n)$	exp	[6]
$F3\|\|f_{max}$	$O^*(n!)$	$O^*(5^n)$	exp	[6]
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$J2\|\|C^k_{max}$	$O^*(2^n)$	$O^*(1.4142^n)$	exp	[7]

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[6] L. Shang, C. Lenté, M. Liedloff, V.T'kindt (2016). An exponential dynamic programming algorithm for the 3-machine flowshop scheduling problem to minimize the makespan, Journal of Scheduling, 21(2), pp.227-233.

[7] F. Della Croce, C. Koulamas, V.T'kindt (2016). A constraint generation approach for two-machine shop problems with jobs selection, Eur. J. Oper. Research, submitted.

- We focus on two technics with applications to scheduling :
  - Branch-and-reduce,
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- What happen when multiple objectives are optimized ?

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Branch-and-... What ? !

- Branch-and-Reduce (BaR) ressembles to known exact algorithms like Branch-and-Bound or Branch-and-Cut...
- A BaR algorithm implements three components :
  - A branching rule,
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- It is different from known branching algorithms,
  - Lower bounding : not a stopping rule,
    ⇒ We cannot prove that in the worst-case it always prune nodes,
  - Dominance conditions : not reduction rules,
    ⇒ They are only sufficient conditions of optimality.

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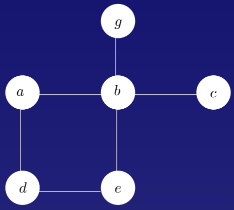
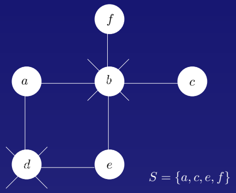
Branch-and-Reduce and the MIS

- Consider the *Maximum Independent Set* (MIS) problem :
  Let $G = (V, E)$ be an undirected graph,
  An independent set S is a set of vertices such that no two vertices from S are connected by an edge,
  The MIS problem consists in finding S with a maximum cardinality,



A graph G



A Maximum Independent Set S

Branch-and-Reduce and the MIS

- A first analysis of the problem shows that when $d(v) \leq 2$, $\forall v \in V$, the problem is polynomially solvable,
- This is used as a stopping rule in the Branch-and-Reduce approach (BraRed),
- We now present the BraRed algorithm,

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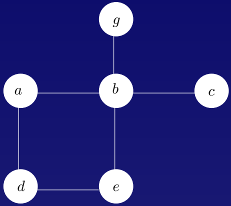
- General case : the maximum degree of vertices is at least 3.



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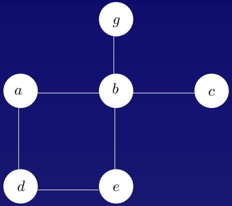

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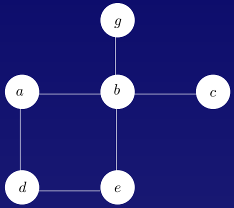

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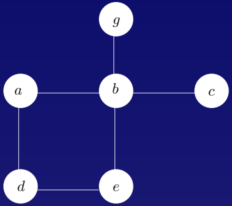

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- General case : the maximum degree of vertices is at least 3.
- Select vertex b of degree 4,
- Case 1 : $b \in S$, then a, c, e and f are removed. Vertex $d \in S$ by deduction.
- Case 2 : $b \notin S$, then c and f have degree 0 and are put in S. Vertices a, d, e form a graph of max degree 2... solvable in polynomial time.

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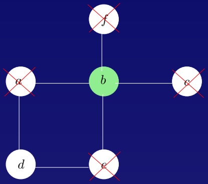

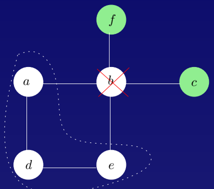

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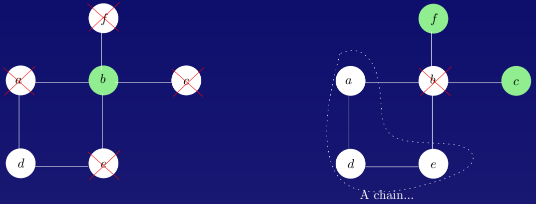

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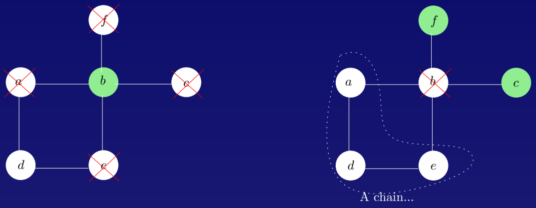

- In that case 2 nodes have been built.
- Reduction rule : when a decision is taken on a vertex v, decisions are taken for all its neighborhood,
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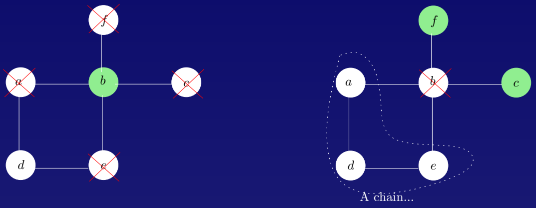

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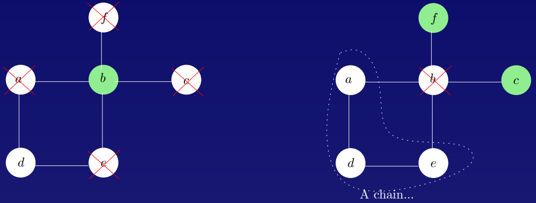

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- The `BraRed` algorithm (main iterated loop) :
  - Put all vertices of degree 0 into S,
  - Let v be the vertex with maximum degree :
    - if $d(v) \geq 3$, create two child nodes : one with $v \in S$, another with $v \notin S$. Propagate to its neighborhood.
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- The above processing is applied on any unbranched node in `BraRed`.

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- What is the worst-case complexity of `BraRed` ?
- Let us observe the branching rule : $T(n)$ is the time required to solve a problem with n vertices,
- We can state that :

$$T(n) \leq T(n-1-d(v)) + T(n-1)$$

with v the vertex selected for branching.

- The worst case is obtained when $d(v)$ is minimal, *i.e.* $d(v) = 3$.
- So, in the worst case the time complexity for solving the problem is $T(n) = T(n-4) + T(n-1)$ with $n = |V|$.

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- What is the worst-case complexity of `BraRed` ?
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- By using a solver like Mathlab (for instance), we obtain $O^*(1.3803^n)$ as the worst-case time complexity for BraRed.
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- Assume that we have a set of K criteria Z_i to minimise over a set of solutions S,
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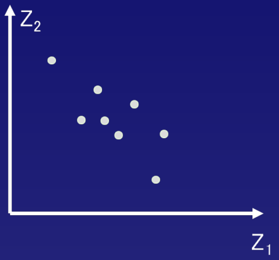

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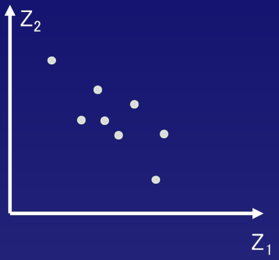

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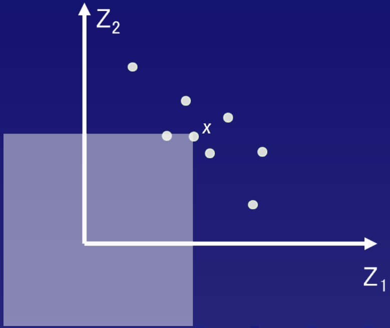

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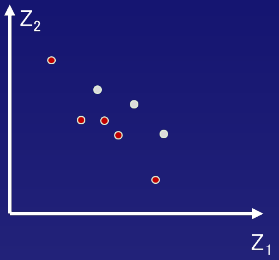

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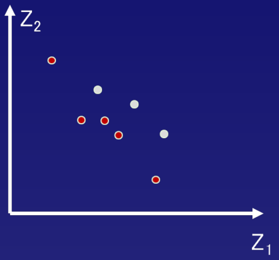

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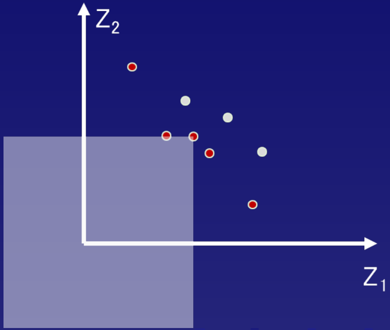

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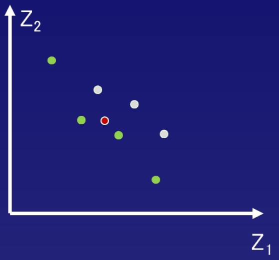

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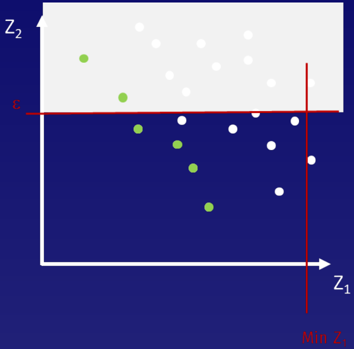

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Branch-and-Reduce and multiobjective scheduling

- Let us consider the following scheduling problem :
  - n jobs are to be scheduled on 2 machines,
  - Each job j is defined by $p_{j,1}$, $p_{j,2}$,
  - All jobs share the same common due date d (unknown),
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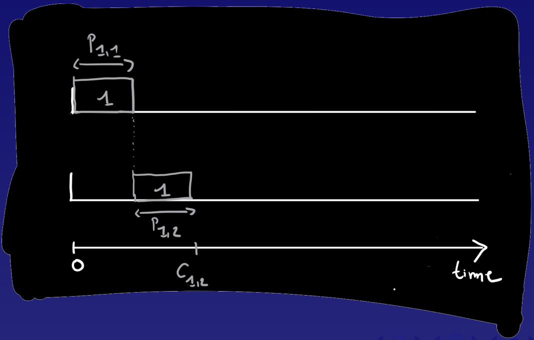

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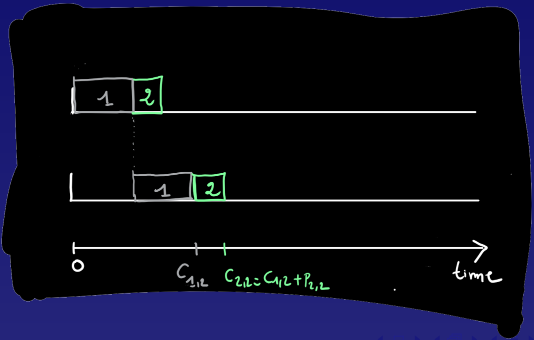

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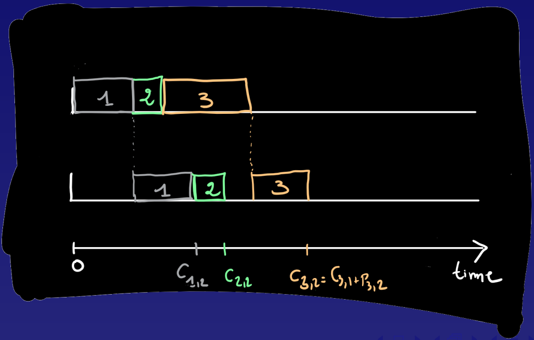

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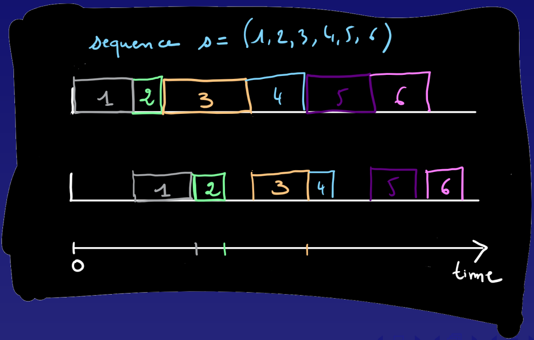

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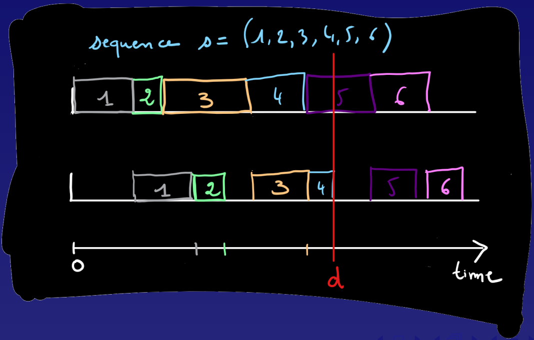

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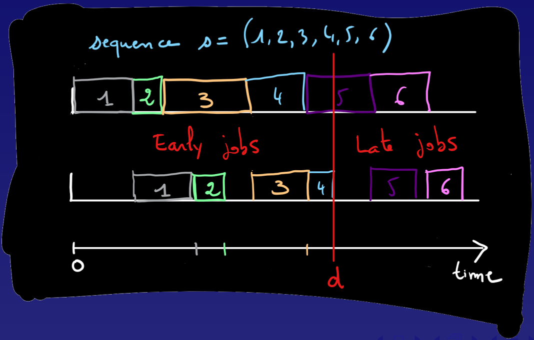

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- This problem is referred to as $F2|d_j = d|d, \sum_j U_j$,
- It is $\mathcal{NP}$-hard in the ordinary sense,
- In brief, what we can show :
  - There are exactly $(n+1)$ non dominated criteria vectors,
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- Let us design the following BraRed algorithm ([7]) :
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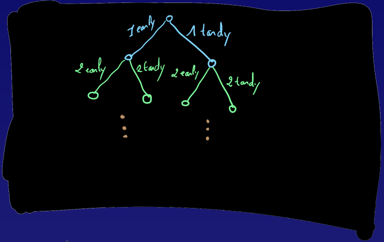

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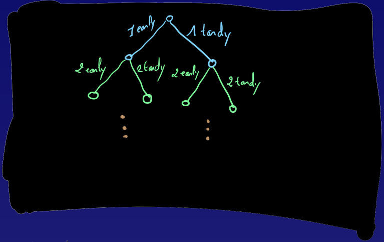

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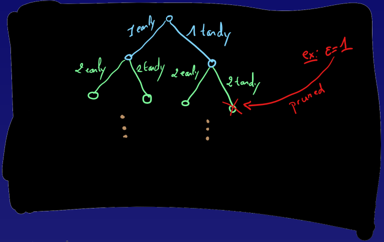

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- Time complexity analysis :
- Due to the branching scheme, we have :

$$T(n,\epsilon) = T(n-1,\epsilon) + T(n-1,\epsilon-1) = \binom{n}{\epsilon}.$$

- Wlg, assume that $\epsilon = \lambda n$ with $\lambda \in [0;1]$,

Theorem ([7])

BraRed solves the problem with a worst-case time complexity in $O^*([(\frac{1}{\lambda})^{\lambda}(\frac{1}{1-\lambda})^{1-\lambda}]^n)$, *i.e.* $O^*(c(\lambda)^n)$ with $c(\lambda) = (\frac{1}{\lambda})^{\lambda}(\frac{1}{1-\lambda})^{1-\lambda}$, and polynomial space.

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- The result is shown by using Stirling's formulae to approximate $k!$,
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4	0.25	1.7547	$O^*(1.7547^n)$
5	0.20	1.6493	$O^*(1.6493^n)$
6	0.16	1.5691	$O^*(1.5691^n)$
7	0.14	1.5069	$O^*(1.5069^n)$
8	0.12	1.4575	$O^*(1.4575^n)$
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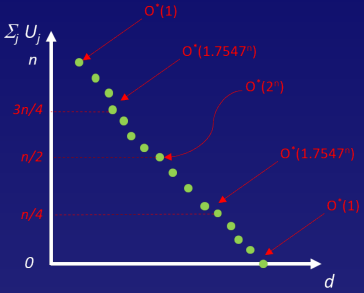
6	0.16	1.5691	$O^*(1.5691^n)$
7	0.14	1.5069	$O^*(1.5069^n)$
8	0.12	1.4575	$O^*(1.4575^n)$
9	0.11	1.4174	$O^*(1.4174^n)$
10	0.10	1.3841	$O^*(1.3841^n)$



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- We have upper bounds on the worst-case time complexity for the flowshop problem,
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- It is an old technique which consists in **sorting** "data" to make the **search** for an optimal solution more efficient,
- It has been proposed by Horowitz and Sahni ([8]) to solve the knapsack problem (SCP),
- It has been extended by Lenté et al. ([4]) to solve *Multiple Constraint Problems* (MCP),

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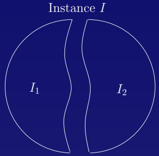

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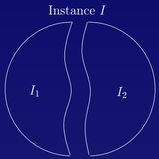

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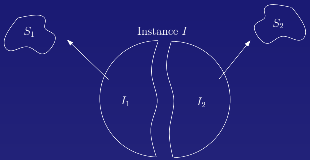

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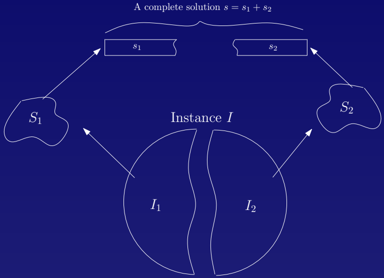
- By recombination of partial solutions, find the optimal solution of the initial problem



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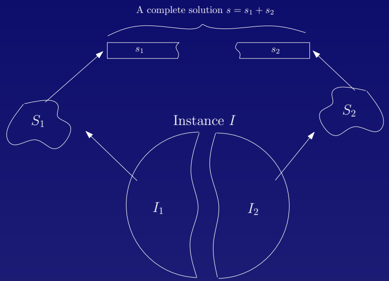

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Sort & Search and multiobjective optimization

- The idea : cut the cake into two equal-size pieces and just pay for one (but take both !),
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- The (MMCP) can be defined as follows :

$$
\begin{aligned}
&\text{Minimize } f_1(\boldsymbol{a_j}, b_k^1)\\
&\ldots\\
&\text{Minimize } f_K(\boldsymbol{a_j}, b_k^K)\\
&\text{s.t.}\\
&\quad g_\ell(\boldsymbol{a_j}, b_k^{K+\ell}) \geq 0, \quad (1 \leq \ell \leq d_B)\\
&\quad \boldsymbol{a}_j \in A,\ \boldsymbol{b}_k \in B.
\end{aligned}
$$

with A a table of n_A vectors of dimension d_A, B a table of n_B vectors of dimension $(d_B + K)$, f_h $(1 \leq h \leq K)$ and g_ℓ $(1 \leq \ell \leq d_B)$, $(d_B + K)$ functions from $\mathbb{R}^{d_A+1}$ to $\mathbb{R}$ which are non-decreasing with respect to their last variable.

Sort & Search and multiobjective optimization

Theorem ([9])

If a multiobjective optimization problem can be reformulated as a (MMCP) then there exists a *Sort & Search* algorithm to solve it and which requires $O(|E| \cdot (K n_B \log_2^{d_B+2}(n_B) + K 2^K))$ time and $O(n_B \log_2^{d_B-1}(n_B) + |E|)$ space.

- Example : the $P2|d_i|C_{\max}, L_{\max}$ problem,
  - $K = 2$, $d_B = 3$, $n_A = n_B = n/2$,
  - Worst-case time complexity is $O^*(|E| \cdot 1.4143^n)$ and the space complexity is $O^*(1.4143^n + |E|)$,
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