

Representation of the non-dominated set of biobjective combinatorial optimization problems

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“*Tractability in multiobjective combinatorial optimization*”

C. Fonseca, L. Paquete, S. Ruzika, B. Schulze, M. Stiglmayr, and D. Willems.
Approximating the Hypervolume by Quadratic Scalarizations. *Work in Progress*, 2016.

D. Vaz, L. Paquete, C. M. Fonseca, K. Klamroth, and M. Stiglmayr. Representation of the
non-dominated set in biobjective combinatorial optimization. *Computers & Operations
Research*, 63:172–186, 2015.

The size of the nondominated set for many discrete multicriteria optimization problems may be *exponentially* large:

- ▶ *Shortest path problem*
(Hansen 1979)
- ▶ *Minimum spanning tree problem*
(Hamacher and Ruhe 1994)
- ▶ *Fixed cardinality problems*
(Ehgott 2001)

The nondominated set of a discrete multicriteria optimization problems may be *too large* in practice. A *representation* of the nondominated set is more appropriate for a decision maker.

- ▶ What *properties* should such a representation have?

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- ▶ How *hard* is to compute a representation?

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A *representation* of the nondominated set is more appropriate for a decision maker.

- ▶ What *properties* should such a representation have?
- ▶ How *hard* is to compute a representation?
- ▶ Can a representation be computed apriori? Or is it based on the knowledge on the complete nondominated set?

Contents

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion



multicriteria optimization problem

$$\max_{x \in X} f(x) = (f^1(x), \dots, f^Q(x))$$

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efficient/nondominated

A solution $x^* \in X$ is *efficient* if there is no $x \in X$ such that

$$f(x) \geq f(x^*)$$

$x^* \in X$ efficient $\implies y = f(x^*)$ *nondominated*.

The set of all nondominated points is the *nondominated set* Y_N .

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The set of all nondominated points is the *nondominated set* Y_N .

We assume that Y_N is also *finite* (but may be still *very large*).

Quality Measures for Representations

Representation of
the non-dominated
set

M. Stiglmayr

Cardinality: contains a reasonable small number of points

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

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Uniformity Problem

Let $Y_N = \{y_1, \dots, y_n\} \subseteq \mathbb{R}^2$ and let $k \in \{1, \dots, n\}$.

Find a subset $R \subseteq Y_N$, $|R| = k$, such that the points in R are as *uniformly spread* as possible (*k-dispersion problem*).

$$\max_{\substack{R \subseteq Y_N \\ |R|=k}} \min_{y_i, y_j \in R} \{\|y_i - y_j\|\}$$

Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

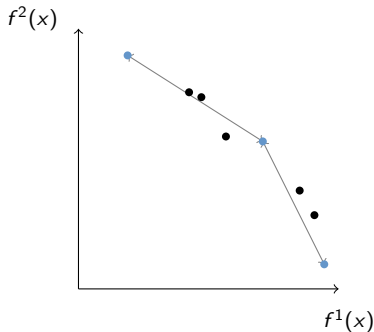
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Representation of
the non-dominated
set

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Coverage Problem

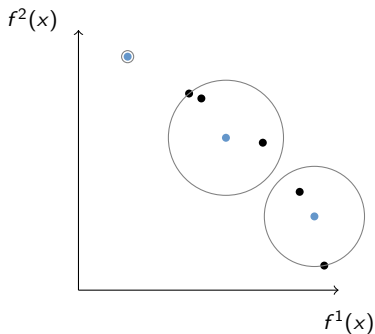
Find a subset $R \subseteq Y_N$, $|R| = k$, such that every point in Y_N is *as close as possible* of a point in R (*k-center problem*).

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Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

ε -indicator Problem

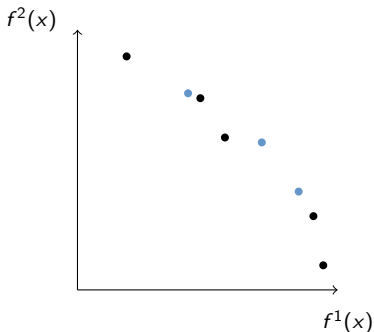
Find a subset $R \subseteq Y_N$, $|R| = k$, such that every point in Y_N is *dominated by a multiple of a point in R* .

$$\min_{\substack{R \subseteq Y_N \\ |R|=k}} \max_{y_i \in Y_N} \min_{y_j \in R} \max \left\{ \frac{y_i^1}{y_j^1}, \frac{y_i^2}{y_j^2} \right\}$$

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Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

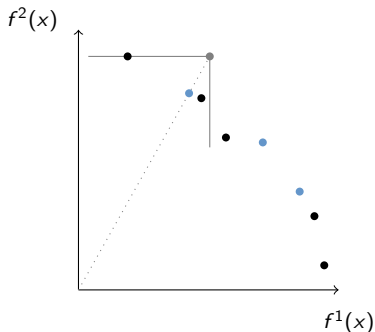
Hypervolume

Conclusion

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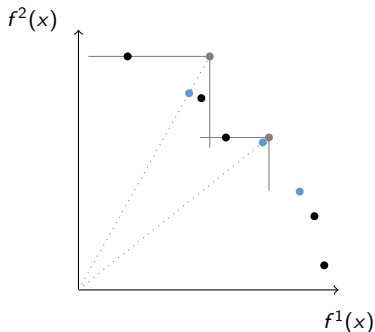
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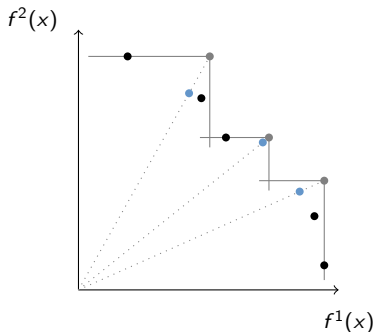
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For *biobjective* problems the following representation problems can be solved in polynomial time:

Find a subset $R \subseteq Y_N$ with cardinality $|R| = k$ that

- ▶ maximizes uniformity
- ▶ minimizes coverage
- ▶ minimizes ε -indicator
- ▶ maximizes uniformity and minimizes coverage, or maximizes uniformity and minimizes ε -indicator, or minimizes coverage and ε -indicator
- ▶ maximizes uniformity and minimizes coverage and ε -indicator

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Proposition

The problem of computing an optimal representative subset (wrt. coverage, uniformity, ε -indicator) is equivalent to a corresponding 1-D location problem with the points in a line.

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

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The problem of computing an optimal representative subset (wrt. coverage, uniformity, ε -indicator) is equivalent to a corresponding 1-D location problem with the points in a line.

Proof (sketch)

- ▶ nondominated points can be totally ordered
- ▶ coverage, uniformity and ε -indicator are monotone on the nondominated set of a biobjective problem

Two Solution Approaches

Two Solution Approaches

Dynamic Programming

The optimal subset of $\{y_j, \dots, y_n\}$ with cardinality i that contains y_j is obtained from:

- ▶ *Optimal solutions of smaller subproblems*: The optimal subset of $\{y_\ell, \dots, y_n\}$ with cardinality $i - 1$ that contains y_ℓ , $\ell > j$
- ▶ *Update value*: A value that depends of y_j and y_ℓ .

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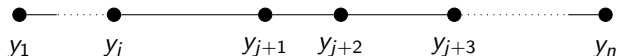
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- ▶ *Update value*: A value that depends of y_j and y_ℓ .

Threshold Algorithm

- ▶ Select a threshold value for a quality measure.
- ▶ Solve a feasibility problem to check whether there exists a subset satisfying this threshold.
- ▶ If the threshold is selected with bisection method it requires the solution of $\mathcal{O}(\log n)$ feasibility problems.

Dynamic Programming for Uniformity

Let $U_{i,j}$ denote the uniformity value of the optimal subset of $\{y_j, \dots, y_n\}$ with cardinality i that contains y_j .



Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

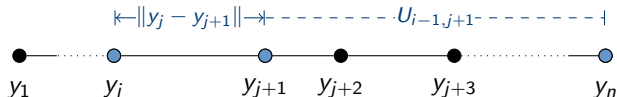
A Priori

Hypervolume

Conclusion

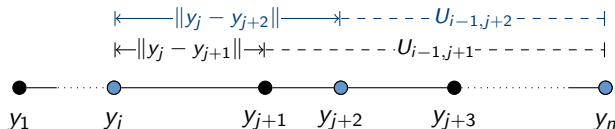
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$$U_{i,j} = \max \left\{ \min \{ \|y_j - y_{j+1}\|, U_{i-1, j+1} \} \right.$$

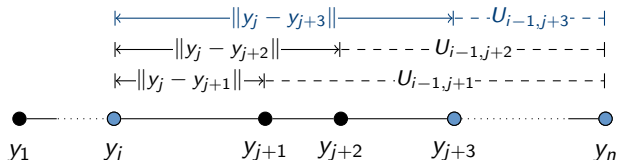
Dynamic Programming for Uniformity



$$U_{i,j} = \max \begin{cases} \min \{ \|y_j - y_{j+1}\|, U_{i-1,j+1} \} \\ \min \{ \|y_j - y_{j+2}\|, U_{i-1,j+2} \} \end{cases}$$

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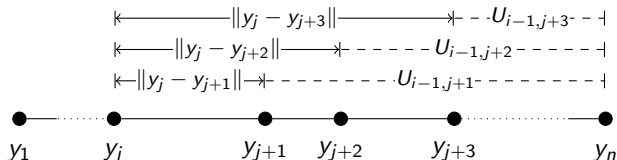
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Recursion

$$U_{1,j} = \infty$$

$$U_{i,j} = \max_{j+1 \leq \ell \leq n-i+2} \{ \min \{ \|y - j - y_\ell\|, U_{i-1,\ell} \} \}$$

Recursion

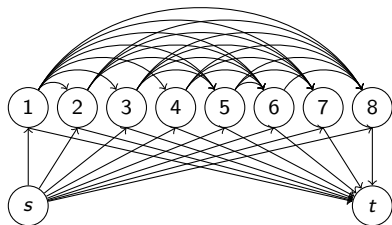
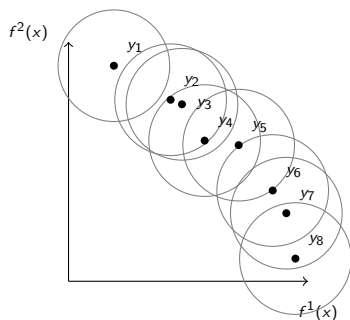
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The optimal uniformity value is given by

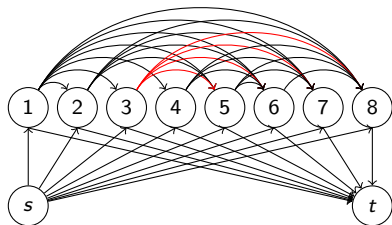
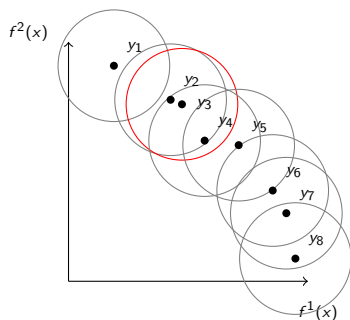
$$\max_{1 \leq \ell \leq n-k+1} \{ U_{k,\ell} \}$$

Threshold Algorithm for Uniformity



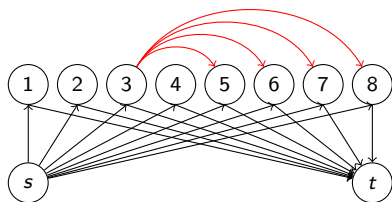
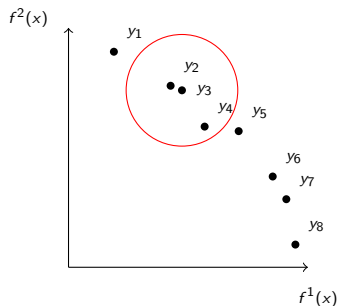
- ▶ $G(U) = (V, E)$ with $V = \{1, \dots, n\} \cup \{s, t\}$ and $E = \{(s, i), (i, t) : i \in \{1, \dots, n\}\} \cup \{(i, j) : \|y_i - y_j\| \geq U\}$

Threshold Algorithm for Uniformity



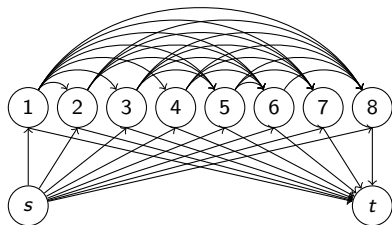
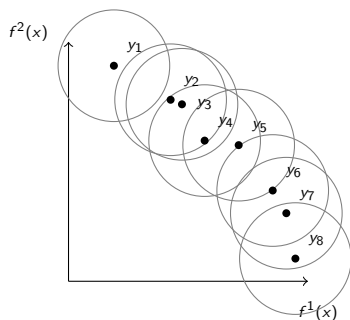
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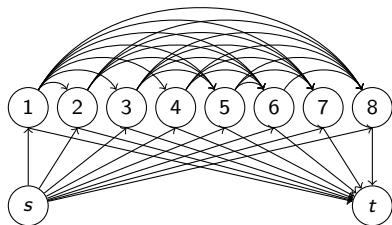
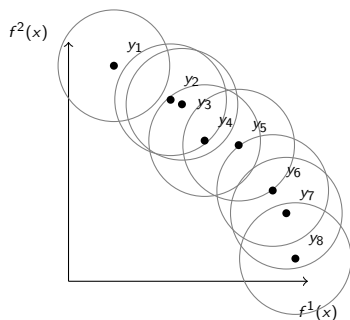
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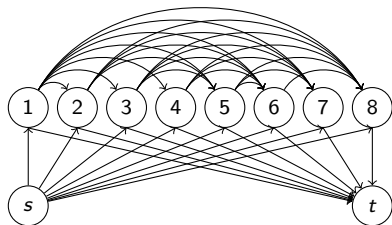
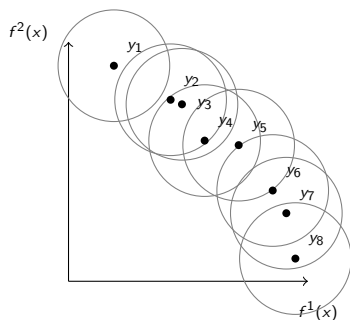
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- ▶ every s - t -path corresponds to subset with uniformity value not smaller than the threshold U

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- ▶ every s - t -path corresponds to subset with uniformity value not smaller than the threshold U
- ▶ if there is a s - t -path of length k in $G(U)$, then there exists for all $k' < k$ a s - t -path of length k'

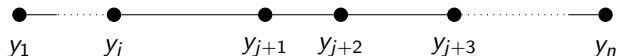
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- ▶ if there is a s - t -path of length k in $G(U)$, then there exists for all $k' < k$ a s - t -path of length k'
- ▶ longest path problem Greedy-solvable in $G(U)$

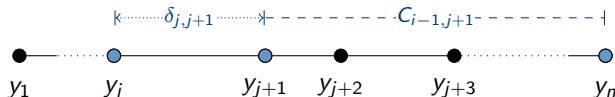
Dynamic Programming for Coverage

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Dynamic Programming for Coverage

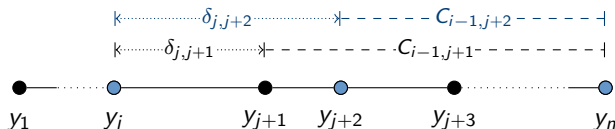
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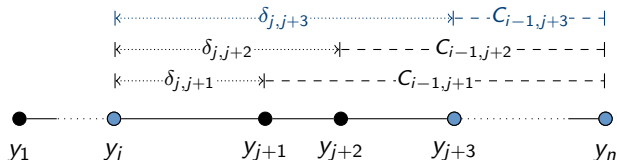
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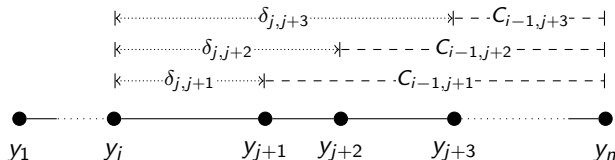
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$$C_{i,j} = \min \begin{cases} \max \{ \delta_{j,j+1}, C_{i-1,j+1} \} \\ \max \{ \delta_{j,j+2}, C_{i-1,j+2} \} \\ \max \{ \delta_{j,j+3}, C_{i-1,j+3} \} \end{cases}$$

Dynamic Programming for Coverage

Let $C_{i,j}$ denote the coverage value of the optimal subset of $\{y_j, \dots, y_n\}$ with cardinality i that contains y_j .

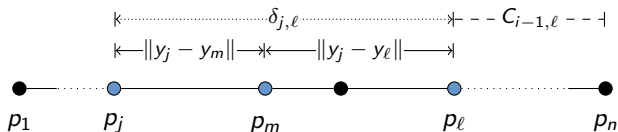


$$C_{i,j} = \min \begin{cases} \max \{ \delta_{j,j+1}, C_{i-1,j+1} \} \\ \max \{ \delta_{j,j+2}, C_{i-1,j+2} \} \\ \max \{ \delta_{j,j+3}, C_{i-1,j+3} \} \\ \vdots \end{cases}$$

Dynamic Programming for Coverage

Let $y_{\bar{m}}$ be the *worst covered point* between y_j and y_ℓ , i. e. the point maximizing the minimum distance to y_j and y_ℓ .

$$\delta_{j,\ell} = \max_{j+1 \leq m \leq \ell-1} \{ \min \{ \|y_j - y_m\|, \|y_m - y_\ell\| \} \}$$



Recursion

$$C_{1,j} = \|y_j - y_n\|$$

$$C_{i,j} = \min_{j+1 \leq l \leq n-i+2} \{\max\{\delta_{j,l}, C_{i-1,l}\}\}$$

Recursion

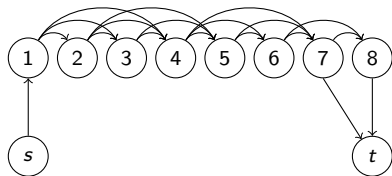
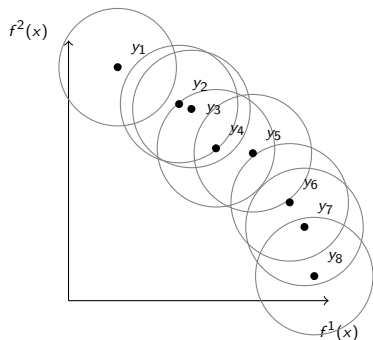
$$C_{1,j} = \|y_j - y_n\|$$

$$C_{i,j} = \min_{j+1 \leq \ell \leq n-i+2} \{\max\{\delta_{j,\ell}, C_{i-1,\ell}\}\}$$

The optimal coverage value is given by

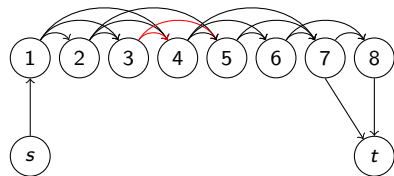
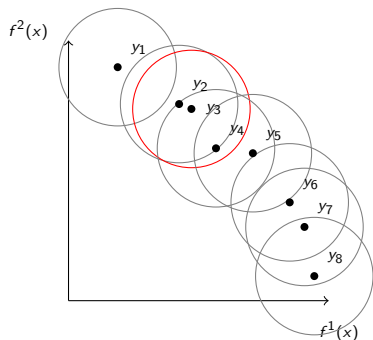
$$\min_{1 \leq j \leq n-k+1} \{\max\{\|y_1 - y_j\|, C_{k,j}\}\}$$

Threshold Algorithm for Coverage



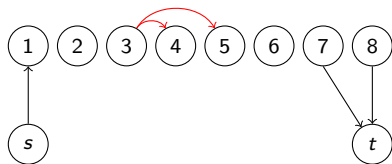
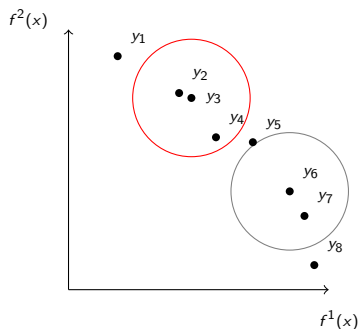
- ▶ $G(C) = (V, E)$ with $V = \{1, \dots, n\} \cup \{s, t\}$ and $E = \{(s, i) : \|y_i - y_1\| \leq C\} \cup \{(j, t) : \|y_j - y_n\| \leq C\} \cup \{(i, j) : \min\{\|y_r - y_i\|, \|y_r - y_j\|\} \leq C \forall r : i < r < j\}$

Threshold Algorithm for Coverage



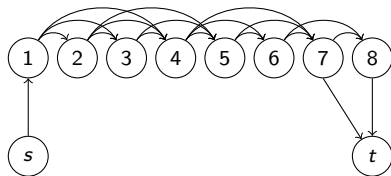
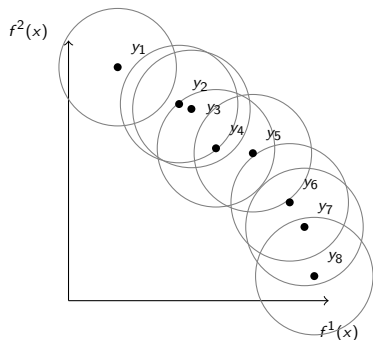
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Threshold Algorithm for Coverage



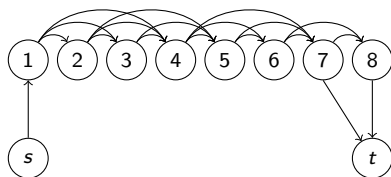
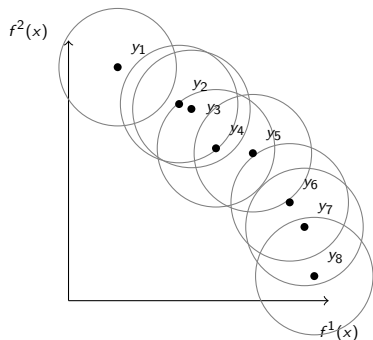
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Threshold Algorithm for Coverage



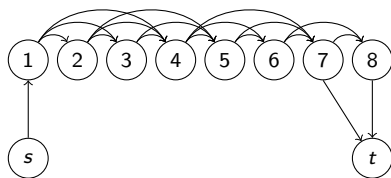
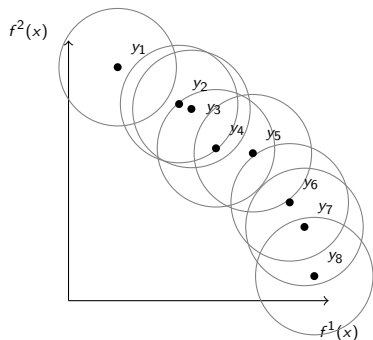
- ▶ $G(C) = (V, E)$
- ▶ every s - t -path corresponds to subset with coverage value not larger than the threshold C

Threshold Algorithm for Coverage



- ▶ $G(C) = (V, E)$
- ▶ every s - t -path corresponds to subset with coverage value not larger than the threshold C
- ▶ if there is a s - t -path of length k in $G(C)$, then there exists for all k' with $k < k' \leq n + 1$ a s - t -path of length k'

Threshold Algorithm for Coverage



- ▶ $G(C) = (V, E)$
- ▶ every s - t -path corresponds to subset with coverage value not larger than the threshold C
- ▶ if there is a s - t -path of length k in $G(C)$, then there exists for all k' with $k < k' \leq n + 1$ a s - t -path of length k'
- ▶ shortest path problem is Greedy-solvable in $G(C)$

Uniformity-coverage Problem

Find subset(s) $R \subseteq Y_N$, $|R| = k$, such that

- ▶ the points in R are as *uniformly spread* as possible
- ▶ every point in Y_N is *as close as possible* of a point in R

$$\max_{\substack{R \subseteq Y_N \\ |R|=k}} \left(\min_{y_i, y_j \in R} \{\|y_i - y_j\|\}, - \max_{y_i \in Y_N} \min_{y_j \in R} \{\|y_i - y_j\|\} \right)$$

Dynamic Programming for Uniformity-coverage

Let $UC_{i,j}$ denote the nondominated uniformity-coverage values of the efficient subsets of $\{y_j, \dots, y_n\}$ with cardinality i that contain y_j .

$$UC_{i,j} = \max \left\{ \begin{array}{l} \left\{ \left(\begin{array}{l} \min \{ \|y_j - y_{j+1}\|, u \} \\ - \max \{ \delta_{j,j+1}, c \} \end{array} \right) : (u, c) \in UC_{i-1,j+1} \right\} \\ \left\{ \left(\begin{array}{l} \min \{ \|y_j - y_{j+2}\|, u \} \\ - \max \{ \delta_{j,j+2}, c \} \end{array} \right) : (u, c) \in UC_{i-1,j+2} \right\} \\ \left\{ \left(\begin{array}{l} \min \{ \|y_j - y_{j+3}\|, u \} \\ - \max \{ \delta_{j,j+3}, c \} \end{array} \right) : (u, c) \in UC_{i-1,j+3} \right\} \\ \vdots \end{array} \right.$$

Dynamic Programming for Uniformity-coverage

Recursion

$$UC_{1,j} = \left(\|y_j - y_n\| \right)^\infty$$

$$UC_{i,j} = \max_{j+1 \leq \ell \leq n-i+2} \left\{ \left(\begin{array}{l} \min \{ \|y_j - y_\ell\|, u \} \\ - \max \{ \delta_{j,\ell}, c \} \end{array} \right) : (u, c) \in UC_{i-1,\ell} \right\}$$

Dynamic Programming for Uniformity-coverage

Recursion

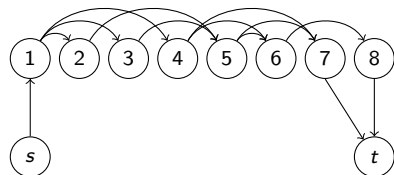
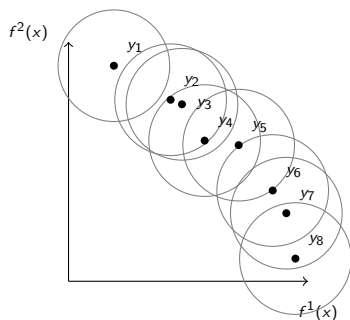
$$UC_{1,j} = \left(\|y_j - y_n\|^\infty \right)$$

$$UC_{i,j} = \max_{j+1 \leq \ell \leq n-i+2} \left\{ \left(\begin{array}{l} \min \{ \|y_j - y_\ell\|, u \} \\ - \max \{ \delta_{j,\ell}, c \} \end{array} \right) : (u, c) \in UC_{i-1,\ell} \right\}$$

The nondominated set is given by

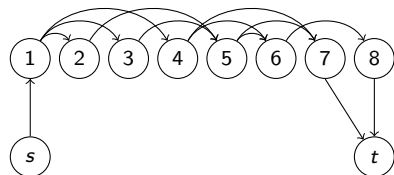
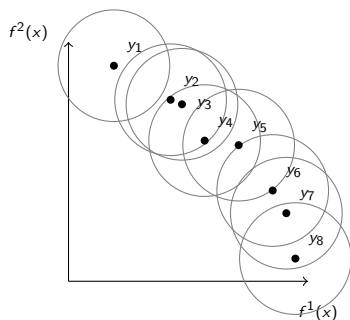
$$\max_{1 \leq j \leq n-k+1} \left\{ \left(\min \{ \|y_1 - y_j\|, u \}, c \right) : (u, c) \in UC_{k,j} \right\}$$

Threshold Algorithm for Uniformity-Coverage



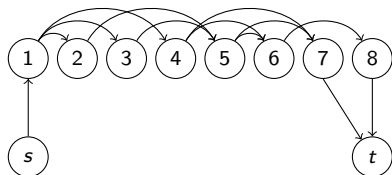
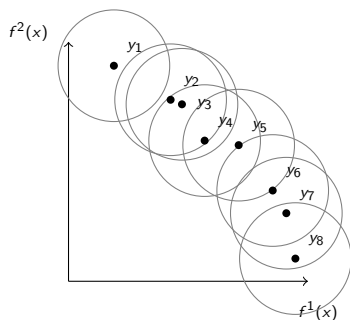
► $G(U, C) = G(U) \cap G(C)$

Threshold Algorithm for Uniformity-Coverage



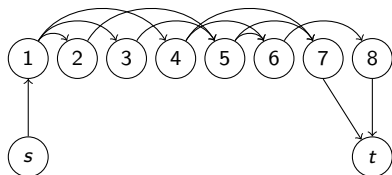
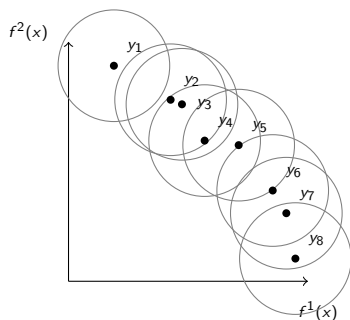
- ▶ $G(U, C) = G(U) \cap G(C)$
- ▶ every s - t -path corresponds to subset with uniformity value not smaller than U and coverage radius not larger than C

Threshold Algorithm for Uniformity-Coverage



- ▶ $G(U, C) = G(U) \cap G(C)$
- ▶ every s - t -path corresponds to subset with uniformity value not smaller than U and coverage radius not larger than C
- ▶ if there are s - t -paths with lengths k^1 and k^2 in $G(U, C)$, then there exists s - t -path of length k' for all $k^1 < k' < k^2$

Threshold Algorithm for Uniformity-Coverage



- ▶ $G(U, C) = G(U) \cap G(C)$
- ▶ every s - t -path corresponds to subset with uniformity value not smaller than U and coverage radius not larger than C
- ▶ if there are s - t -paths with lengths k^1 and k^2 in $G(U, C)$, then there exists s - t -path of length k' for all $k^1 < k' < k^2$
- ▶ compute longest and shortest s - t -path $G(U, C)$

Time complexity

	dynamic programming	threshold algorithm
uniformity	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
coverage	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
ε -indicator	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
unif – cov	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^4)$
ε – unif	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^4)$
cov – ε	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^3 \log n)$
cov – ε – unif	$\mathcal{O}(kn^6 \log n)$	$\mathcal{O}(n^6)$

Computing an Optimal Representation A Priori

Representation of
the non-dominated
set

M. Stiglmayr

The selection of a representing subset *after the computation of the complete nondominated set* seems to be not very efficient.

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

A Priori Computation of Representations

- ▶ Linear continuous case: projection based method wrt. coverage and uniformity (Shao and Ehrgott, 2016)

L. Shao and M. Ehrgott. Discrete representation of non-dominated sets in multi-objective linear programming. *EJOR*, 2016. doi: <http://dx.doi.org/10.1016/j.ejor.2016.05.001>.

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

A Priori Computation of Representations

- ▶ Linear continuous case: projection based method wrt. coverage and uniformity (Shao and Ehrgott, 2016)
- ▶ In the discrete case representation problems involving coverage (and uniformity) are particularly difficult.

L. Shao and M. Ehrgott. Discrete representation of non-dominated sets in multi-objective linear programming. *EJOR*, 2016. doi: <http://dx.doi.org/10.1016/j.ejor.2016.05.001>.

Hypervolume Problem

$Y_N = \{y_1, \dots, y_n\} \subseteq \mathbb{R}^2$ and let $k \in \{1, \dots, n\}$.

Find a subset $R \subseteq Y_N$, $|R| = k$, such that the area dominated by the points in R is maximized.

Hypervolume Problem

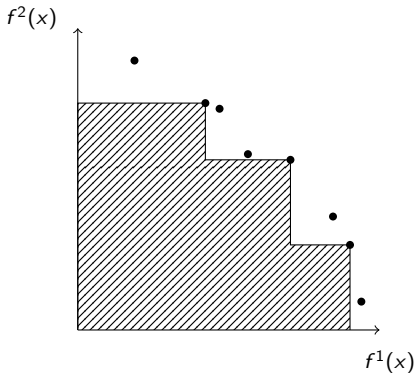
$Y_N = \{y_1, \dots, y_n\} \subseteq \mathbb{R}^2$ and let $k \in \{1, \dots, n\}$.

Find a subset $R \subseteq Y_N$, $|R| = k$, such that the area dominated by the points in R is maximized.

i. e. for a maximization problem
(with $p \geq 0 \quad \forall y \in Y_N$):

$$\max_{x \in X} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

$$\text{Vol} \left(\bigcup_{y \in R} (y - \mathbb{R}_+^Q) \cap \mathbb{R}_+^Q \right)$$



Find one Point Maximizing the Hypervolume

We consider a integer linear biobjective maximization problem ($a, b \geq 0$):

$$\max_{x \in X} \begin{pmatrix} a^\top x \\ b^\top x \end{pmatrix}$$

Representation of
the non-dominated
set

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

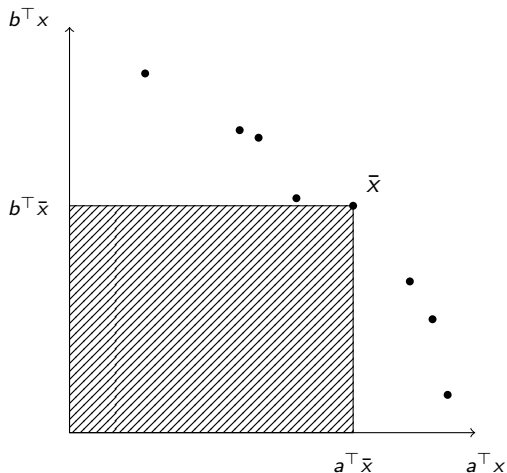
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Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Find one Point Maximizing the Hypervolume

The solution $\bar{x} \in X$ maximizing the hypervolume can be computed by

$$\max_{x \in X} x^T a b^T x \quad (*)$$

- optimization of a quadratic objective function over the same constraint set

Find one Point Maximizing the Hypervolume

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- ▶ quadratic scalarization of the biobjective problem

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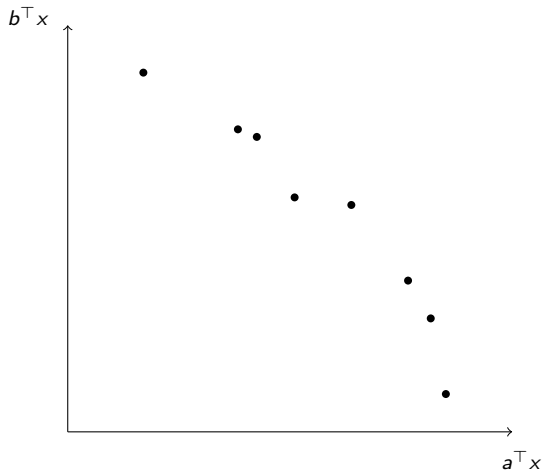
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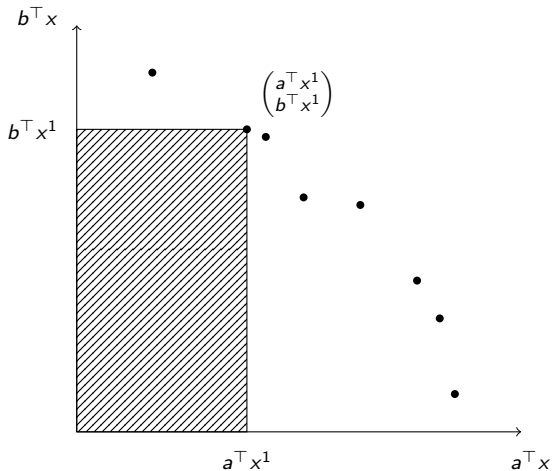
$$\max_{x \in X} x^T a b^T x \quad (*)$$

- ▶ optimization of a quadratic objective function over the same constraint set
- ▶ quadratic scalarization of the biobjective problem
 - ▶ every solution of (*) is efficient for the biobjective problem
 - ▶ by shifting the reference point of the hypervolume every nondominated point can be generated by (*)

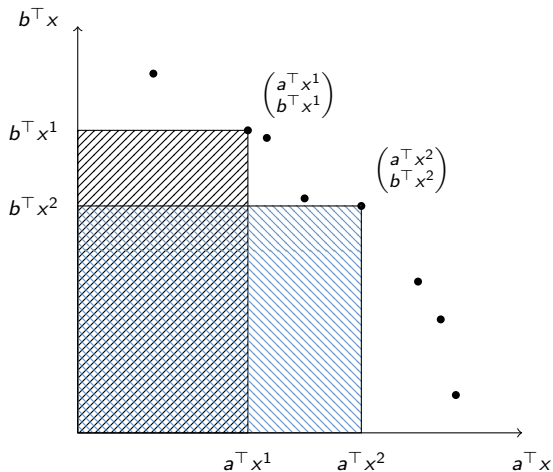
Computing an Optimal Representation wrt. Hypervolume – A Priori



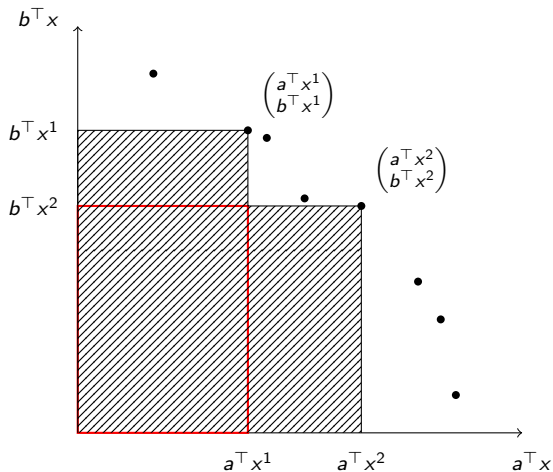
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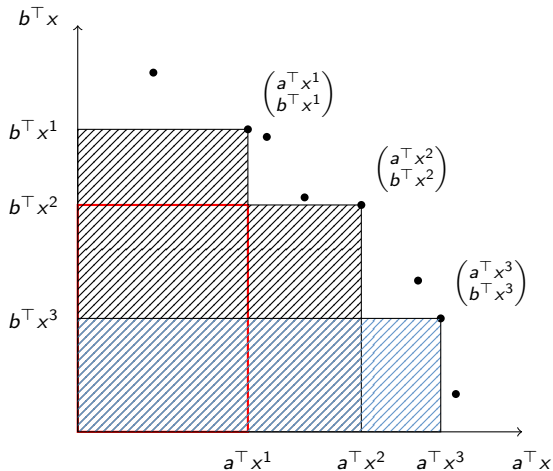
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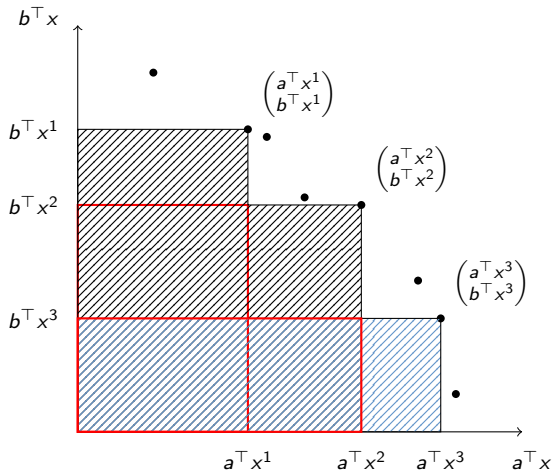
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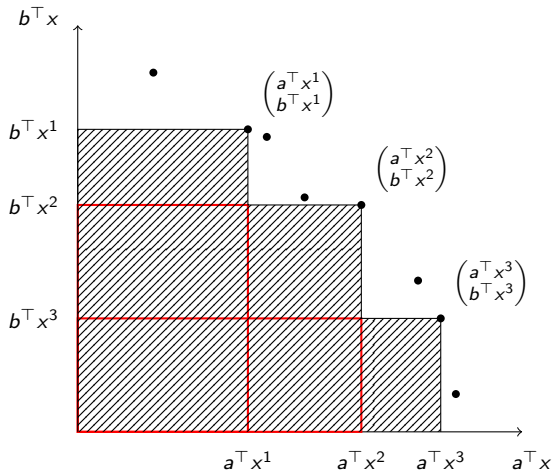
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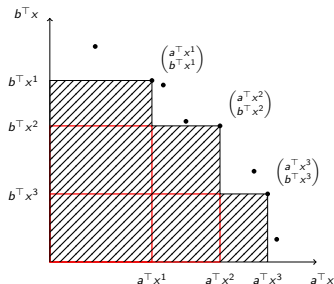


Computing an Optimal Representation wrt. Hypervolume – A Priori



Computing an Optimal Representation wrt. Hypervolume – A Priori

$$V(x_1, x_2) = x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - x_1^\top ab^\top x_2$$

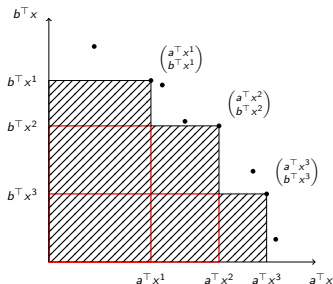


Computing an Optimal Representation wrt. Hypervolume – A Priori

$$V(x_1, x_2) = x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - x_1^\top ab^\top x_2$$

$$V(x_1, x_2, x_3) = V(x_1, x_2) + x_3^\top ab^\top x_3 - x_2^\top ab^\top x_3$$

$$= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 + x_3^\top ab^\top x_3 - x_1^\top ab^\top x_2 - x_2^\top ab^\top x_3$$



Computing an Optimal Representation wrt. Hypervolume – A Priori

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$$= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 + x_3^\top ab^\top x_3 - x_1^\top ab^\top x_2 - x_2^\top ab^\top x_3$$

$$\max \mathbf{x}^\top C \mathbf{x}$$

$$\text{s.t. } a^\top x_i \leq a^\top x_{i+1} \quad \forall i \in \{1, \dots, k-1\}$$

$$x_i \in X \quad \forall i \in \{1, \dots, k\}$$

$$\mathbf{x} = (x_1^\top, x_2^\top, \dots, x_k^\top)^\top$$

Computing an Optimal Representation wrt. Hypervolume – A Priori

$$V(x_1, x_2) = x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - x_1^\top ab^\top x_2$$

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$$= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 + x_3^\top ab^\top x_3 - x_1^\top ab^\top x_2 - x_2^\top ab^\top x_3$$

$$\max \mathbf{x}^\top C \mathbf{x}$$

$$\text{s.t. } a^\top x_i \leq a^\top x_{i+1} \quad \forall i \in \{1, \dots, k-1\}$$

$$x_i \in X \quad \forall i \in \{1, \dots, k\}$$

$$\mathbf{x} = (x_1^\top, x_2^\top, \dots, x_k^\top)^\top$$

with

$$C = \begin{pmatrix} \boxed{ab^\top} & \boxed{-ab^\top} & & & \\ & \boxed{ab^\top} & \boxed{-ab^\top} & & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

Approaches to Use Hypervolume Representation

Representation of
the non-dominated
set

M. Stiglmayr

Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

- ▶ very hard for discrete biobjective optimization problems

Approaches to Use Hypervolume Representation

- ▶ very hard for discrete biobjective optimization problems
- ▶ the special problem structure can be used in approximation schemes

Representation of
the non-dominated
set

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Approaches to Use Hypervolume Representation

- ▶ very hard for discrete biobjective optimization problems
- ▶ the special problem structure can be used in approximation schemes
- ▶ directly applicable to continuous linear biobjective problems

Representation of
the non-dominated
set

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Conclusions and Future Work

- ▶ Representation problems can be solved efficiently a posteriori for biobjective problems.

Representation of
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set

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Introduction

Notation

Quality Measures

A Posteriori

Uniformity

Coverage

Uniformity-Coverage

A Priori

Hypervolume

Conclusion

Conclusions and Future Work

- ▶ Representation problems can be solved efficiently a posteriori for biobjective problems.
- ▶ A priori representation problems wrt. hypervolume can be formulated as quadratic optimization problems.

Conclusions and Future Work

- ▶ Representation problems can be solved efficiently a posteriori for biobjective problems.
- ▶ A priori representation problems wrt. hypervolume can be formulated as quadratic optimization problems.
- ▶ Open topic: Optimal representations for discrete multiobjective optimization problems with $Q > 2$.

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