Representation of the non-dominated set of biobjective combinatorial optimization problems

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"Tractability in multiobjective combinatorial optimization"

C. Fonseca, L. Paquete, S. Ruzika, B. Schulze, M. Stiglmayr, and D. Willems. Approximating the Hypervolume by Quadratic Scalarizations. *Work in Progress*, 2016.

D. Vaz, L. Paquete, C. M. Fonseca, K. Klamroth, and M. Stiglmayr. Representation of the non-dominated set in biobjective combinatorial optimization. *Computers & Operations Research*, 63:172–186, 2015.

Introduction

The size of the nondominated set for many discrete multicriteria optimization problems may be *exponentially* large:

- *Shortest path problem*
  (Hansen 1979)
- *Minimum spanning tree problem*
  (Hamacher and Ruhe 1994)
- *Fixed cardinality problems*
  (Ehgott 2001)

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A *representation* of the nondominated set is more appropriate for a decision maker.

- What *properties* should such a representation have?

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- How *hard* is to compute a representation?

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A *representation* of the nondominated set is more appropriate for a decision maker.

- What *properties* should such a representation have?
- How *hard* is to compute a representation?
- Can a representation be computed apriori? Or is it based on the knowledge on the complete nondominated set?

Contents

Notation

multicriteria optimization problem

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efficient/nondominated

A solution $x^* \in X$ is *efficient* if there is no $x \in X$ such that

$$f(x) \geq f(x^*)$$

$x^* \in X$ efficient $\implies$ $y = f(x^*)$ *nondominated*.
The set of all nondominated points is the *nondominated set* Y_N.

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We assume that Y_N is also *finite* (but may be still very large).

Quality Measures for Representations

Cardinality: contains a reasonable small number of points

S. Sayın. Measuring the quality of discrete representations of efficient sets in multiple objective mathematical programming. *Mathematical Programming*, 87(3):543–560, 2000.

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ε-indicator: each nondominated point is close to a point in the representative subset wrt. a multiplicative distance

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Uniformity Problem

Let $Y_N = \{y_1, \dots, y_n\} \subseteq \mathbb{R}^2$ and let $k \in \{1, \dots, n\}$.
Find a subset $R \subseteq Y_N$, $|R| = k$, such that the points in R are as *uniformly spread* as possible (*k-dispersion problem*).

$$\max_{\substack{R \subseteq Y_N \\ |R|=k}} \min_{y_i, y_j \in R} \{\|y_i - y_j\|\}$$

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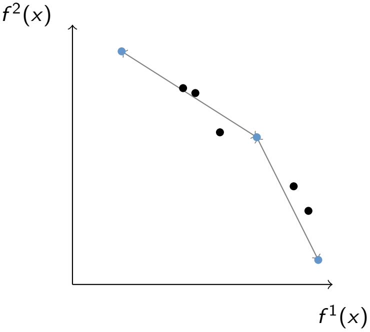

Coverage Problem

Find a subset $R \subseteq Y_N$, $|R| = k$, such that every point in Y_N is *as close as possible* of a point in R (*k-center problem*).

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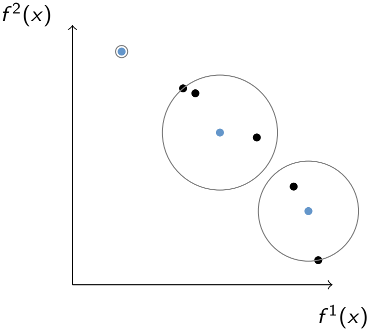

ε-indicator Problem

Find a subset $R \subseteq Y_N$, $|R| = k$, such that every point in Y_N is *dominated by a multiple of a point in R.*

$$\min_{\substack{R \subseteq Y_N \\ |R|=k}} \max_{y_i \in Y_N} \min_{y_j \in R} \max \left\{ \frac{y_i^1}{y_j^1}, \frac{y_i^2}{y_j^2} \right\}$$

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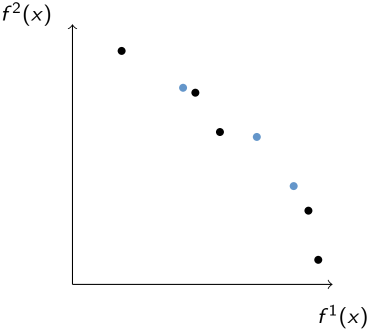

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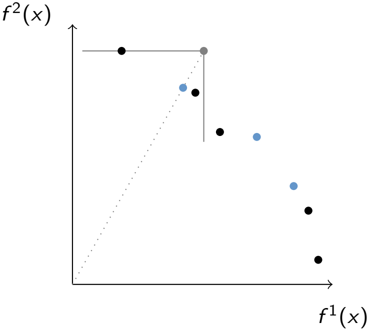

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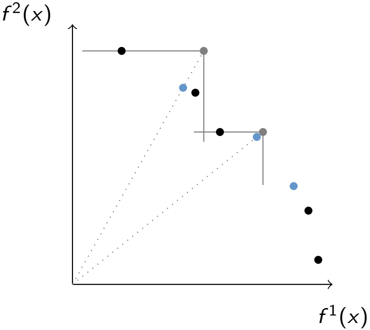

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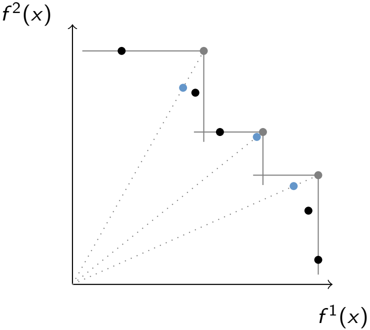

For *biobjective* problems the following representation problems can be solved in polynomial time:

Find a subset $R \subseteq Y_N$ with cardinality $|R| = k$ that

- maximizes uniformity
- minimizes coverage
- minimizes ε-indicator
- maximizes uniformity and minimizes coverage, or maximizes uniformity and minimizes ε-indicator, or minimizes coverage and ε-indicator
- maximizes uniformity and minimizes coverage and ε-indicator

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Proposition

The problem of computing an optimal representative subset (wrt. coverage, uniformity, ε-indicator) is equivalent to a corresponding 1-D location problem with the points in a line.

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Proof (sketch)

- nondominated points can be totally ordered
- coverage, uniformity and ε-indicator are monotone on the nondominated set of a biobjective problem

Two Solution Approaches

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Dynamic Programming

The optimal subset of $\{y_j, \ldots, y_n\}$ with cardinality i that contains y_j is obtained from:

- *Optimal solutions of smaller subproblems*: The optimal subset of $\{y_\ell, \ldots, y_n\}$ with cardinality $i-1$ that contains y_ℓ, $\ell > j$
- *Update value*: A value that depends of y_j and y_ℓ.

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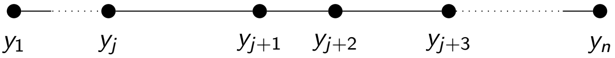
Threshold Algorithm

- Select a threshold value for a quality measure.
- Solve a feasibility problem to check weather there exists a subset satisfying this threshold.
- If the threshold is selected with bisection method it requires the solution of $\mathcal{O}(\log n)$ feasibility problems.

Dynamic Programming for Uniformity

Let $U_{i,j}$ denote the uniformity value of the optimal subset of $\{y_j, \dots, y_n\}$ with cardinality i that contains y_j.



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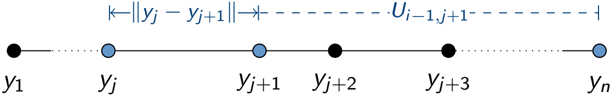

$$U_{i,j} = \max \begin{cases} \min \left\{ \|y_j - y_{j+1}\|, U_{i-1,j+1} \right\} \\ \\ \\ \end{cases}$$

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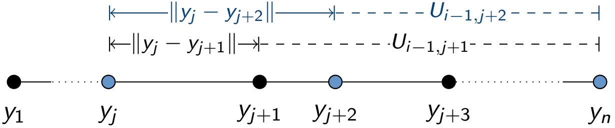

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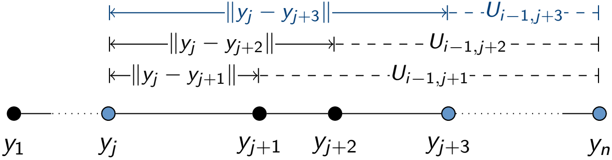

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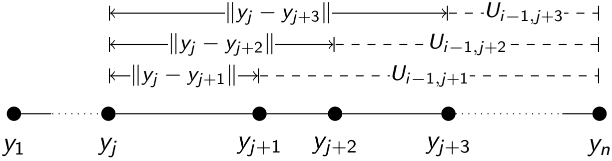

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Dynamic Programming for Uniformity

Recursion

$$U_{1,j} = \infty$$
$$U_{i,j} = \max_{j+1 \leq \ell \leq n-i+2} \{\min\{\|y - j - y_\ell\|,\ U_{i-1,\ell}\}\}$$

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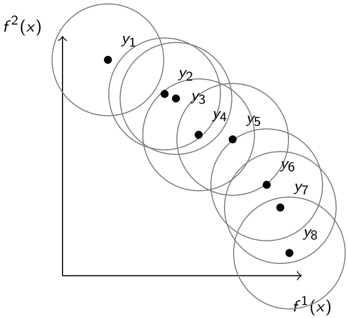
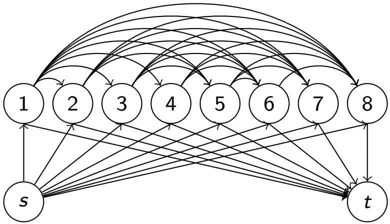
The optimal uniformity value is given by

$$\max_{1 \leq \ell \leq n-k+1} \{U_{k,\ell}\}$$

Threshold Algorithm for Uniformity





- $G(U) = (V, E)$ with $V = \{1, \ldots, n\} \cup \{s, t\}$ and $E = \{(s, i), (i, t) : \ i \in \{1, \ldots, n\}\} \cup \{(i, j) : \|y_i - y_j\| \geq U\}$

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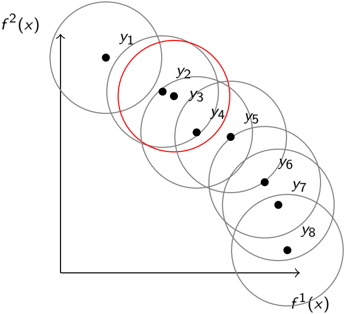

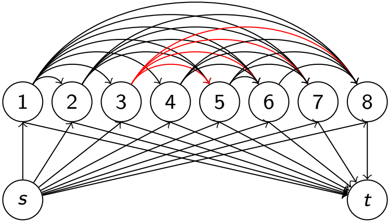

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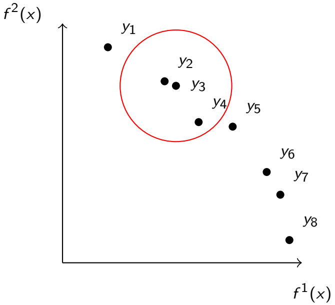

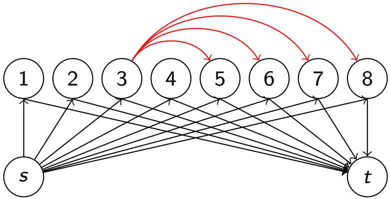

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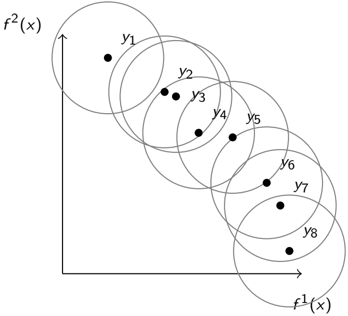

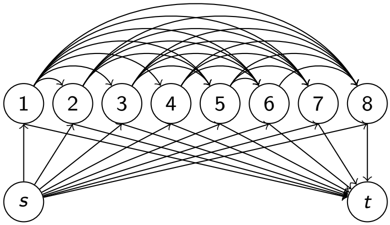

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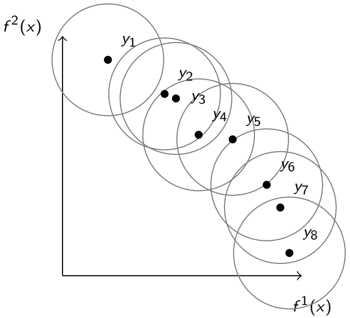
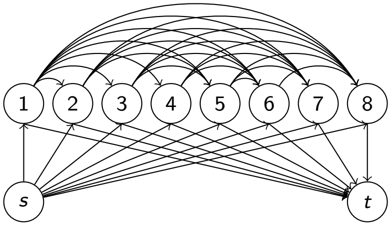
- every s-t-path corresponds to subset with uniformity value not smaller than the threshold U

Threshold Algorithm for Uniformity





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- if there is a s-t-path of length k in $G(U)$, then there exists for all $k' < k$ a s-t-path of length k'

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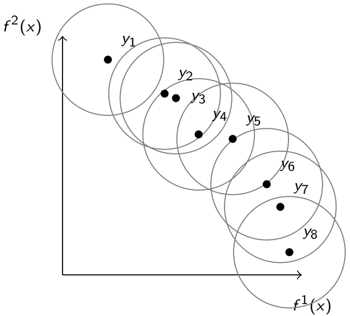

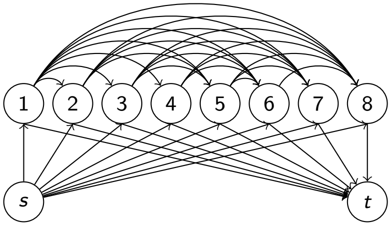

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- every s-t-path corresponds to subset with uniformity value not smaller than the threshold U
- if there is a s-t-path of length k in $G(U)$, then there exists for all $k' < k$ a s-t-path of length k'
- longest path problem Greedy-solvable in $G(U)$

Dynamic Programming for Coverage

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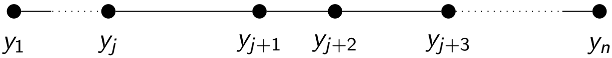

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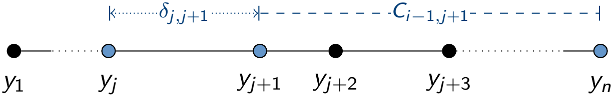

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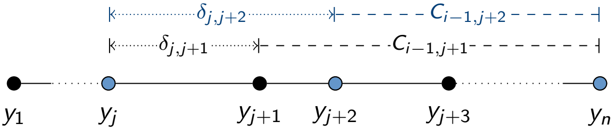

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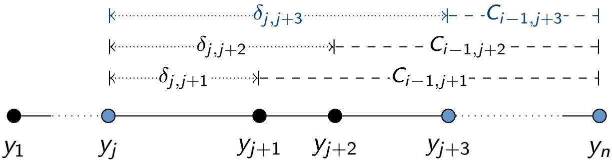
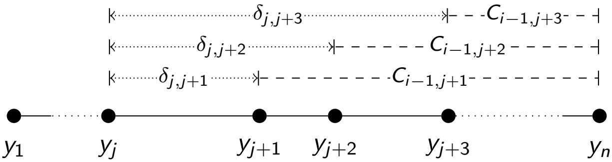


$$C_{i,j} = \min \begin{cases} \max\left\{\delta_{j,j+1},\ C_{i-1,j+1}\right\} \\ \max\left\{\delta_{j,j+2},\ C_{i-1,j+2}\right\} \\ \max\left\{\delta_{j,j+3},\ C_{i-1,j+3}\right\} \\ \end{cases}$$

Dynamic Programming for Coverage

Let $C_{i,j}$ denote the coverage value of the optimal subset of $\{y_j, \ldots, y_n\}$ with cardinality i that contains y_j.



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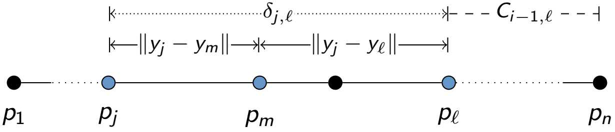
Dynamic Programming for Coverage

Let $y_{\bar{m}}$ be the *worst covered point* between y_j and y_ℓ, i. e. the point maximizing the minimum distance to y_j and y_ℓ.

$$\delta_{j,\ell} = \max_{j+1 \leq m \leq \ell-1} \{\min\{\|y_j - y_m\|,\ \|y_m - y_\ell\|\}\}$$



Dynamic Programming for Coverage

Recursion

$$C_{1,j} = \|y_j - y_n\|$$

$$C_{i,j} = \min_{j+1 \leq \ell \leq n-i+2} \{\max\{\delta_{j,\ell},\ C_{i-1,\ell}\}\}$$

Dynamic Programming for Coverage

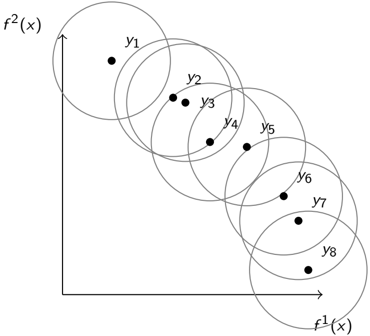
Recursion

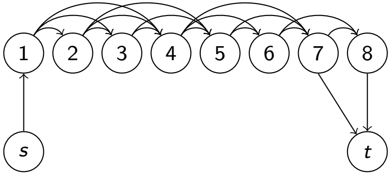
$$C_{1,j} = \|y_j - y_n\|$$
$$C_{i,j} = \min_{j+1 \le \ell \le n-i+2} \{\max\{\delta_{j,\ell},\ C_{i-1,\ell}\}\}$$

The optimal coverage value is given by

$$\min_{1 \le j \le n-k+1} \{\max\{\|y_1 - y_j\|,\ C_{k,j}\}\}$$

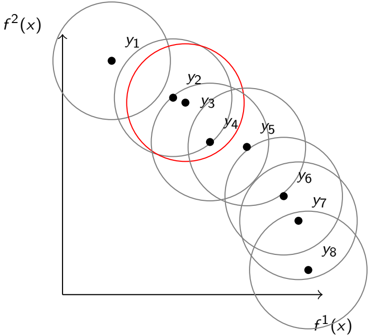
Threshold Algorithm for Coverage

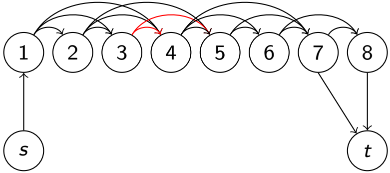




- $G(C) = (V, E)$ with $V = \{1, \dots, n\} \cup \{s, t\}$ and $E = \{(s, i) : \ \|y_i - y_1\| \leq C\} \cup \{(j, t) : \ \|y_j - y_n\| \leq C\} \cup \{(i, j) : \min\{\|y_r - y_i\|, \ \|y_r - y_i\|\} \leq C \, \forall r : i < r < j\}$

Threshold Algorithm for Coverage





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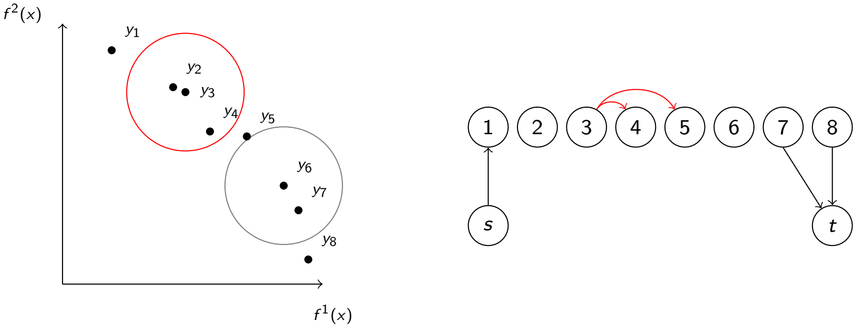

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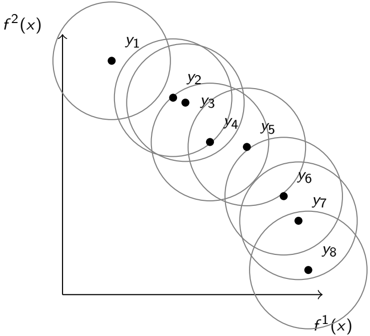

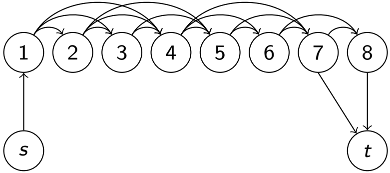

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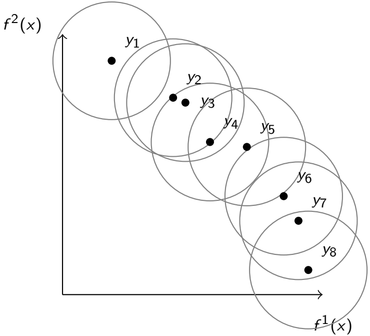
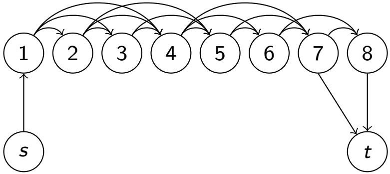
Threshold Algorithm for Coverage





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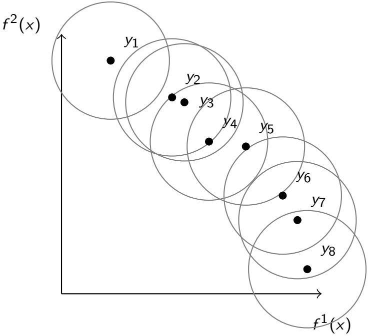
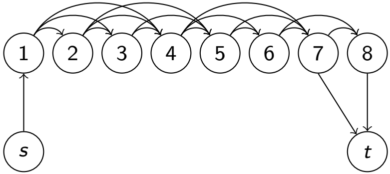
Threshold Algorithm for Coverage





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- shortest path problem is Greedy-solvable in $G(C)$

Dynamic Programming for Uniformity-coverage

Uniformity-coverage Problem

Find subset(s) $R \subseteq Y_N$, $|R| = k$, such that

- the points in R are as *uniformly spread* as possible
- every point in Y_N is *as close as possible* of a point in R

$$\max_{\substack{R \subseteq Y_N \\ |R|=k}} \left(\min_{y_i, y_j \in R} \{\|y_i - y_j\|\}, - \max_{y_i \in Y_N} \min_{y_j \in R} \{\|y_i - y_j\|\} \right)$$

Dynamic Programming for Uniformity-coverage

Let $UC_{i,j}$ denote the nondominated uniformity-coverage values of the efficient subsets of $\{y_j, \ldots, y_n\}$ with cardinality i that contain y_j.

$$UC_{i,j} = \max \begin{cases} \left\{ \begin{pmatrix} \min\{\|y_j - y_{j+1}\|,\ u\} \\ -\max\{\delta_{j,j+1}, c\} \end{pmatrix} : (u,c) \in UC_{i-1,j+1} \right\} \\ \left\{ \begin{pmatrix} \min\{\|y_j - y_{j+2}\|,\ u\} \\ -\max\{\delta_{j,j+2}, c\} \end{pmatrix} : (u,c) \in UC_{i-1,j+2} \right\} \\ \left\{ \begin{pmatrix} \min\{\|y_j - y_{j+3}\|,\ u\} \\ -\max\{\delta_{j,j+3}, c\} \end{pmatrix} : (u,c) \in UC_{i-1,j+3} \right\} \\ \quad \vdots \end{cases}$$

Dynamic Programming for Uniformity-coverage

Recursion

$$UC_{1,j} = \binom{\infty}{\|y_j - y_n\|}$$

$$UC_{i,j} = \max_{j+1 \leq \ell \leq n-i+2} \left\{ \begin{pmatrix} \min\{\|y_j - y_\ell\|,\ u\} \\ -\max\{\delta_{j,\ell},\ c\} \end{pmatrix} :\ (u,c) \in UC_{i-1,\ell} \right\}$$

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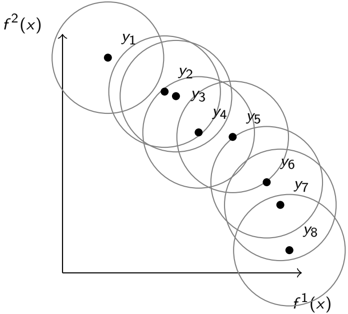
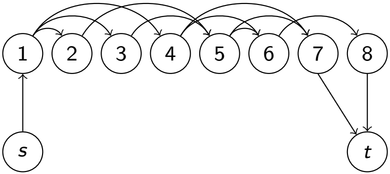
The nondominated set is given by

$$\max_{1 \leq j \leq n-k+1} \left\{ \binom{u}{\min\{\|y_1 - y_j\|,\ c\}} :\ (u, c) \in UC_{k,j} \right\}$$

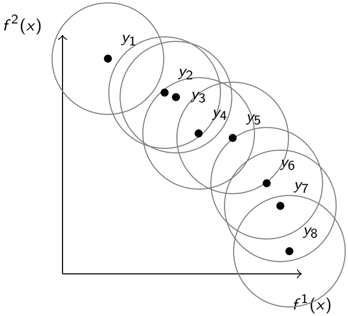
Threshold Algorithm for Uniformity-Coverage

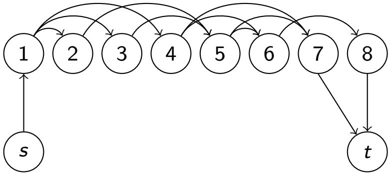




- $G(U, C) = G(U) \cap G(C)$

Threshold Algorithm for Uniformity-Coverage





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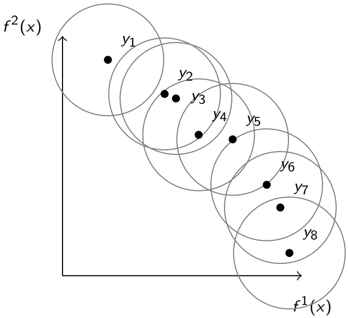
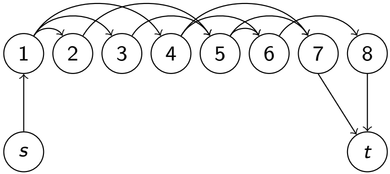
Threshold Algorithm for Uniformity-Coverage





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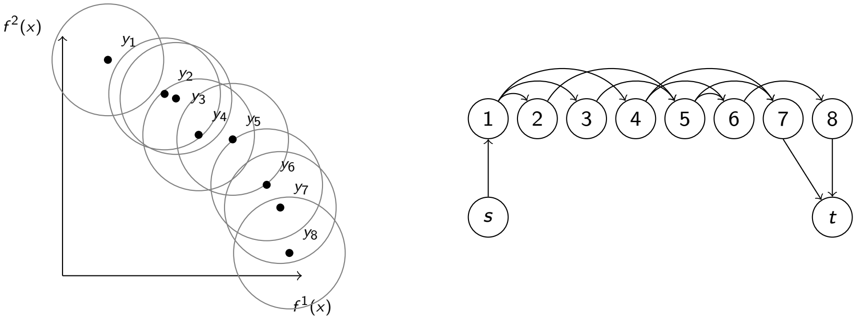
Threshold Algorithm for Uniformity-Coverage



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- if there are s-t-paths with lengths k^1 and k^2 in $G(U, C)$, then there exists s-t-path of length k' for all $k^1 < k' < k^2$
- compute longest and shortest s-t-path $G(U, C)$

Time complexity

	dynamic programming	threshold algorithm
uniformity	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
coverage	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
ε-indicator	$\mathcal{O}(kn + n \log n)$	$\mathcal{O}(n^2 \log n)$
unif – cov	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^4)$
ε – unif	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^4)$
cov – ε	$\mathcal{O}(kn^4 \log n)$	$\mathcal{O}(n^3 \log n)$
cov – ε – unif	$\mathcal{O}(kn^6 \log n)$	$\mathcal{O}(n^6)$

Computing an Optimal Representation A Priori

The selection of a representing subset *after the computation of the complete nondominated set* seems to be not very efficient.

A Priori Computation of Representations

- Linear continuous case: projection based method wrt. coverage and uniformity (Shao and Ehrgott, 2016)

L. Shao and M. Ehrgott. Discrete representation of non-dominated sets in multi-objective linear programming. *EJOR*, 2016. doi: http://dx.doi.org/10.1016/j.ejor.2016.05.001.

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A Priori Computation of Representations

- Linear continuous case: projection based method wrt. coverage and uniformity (Shao and Ehrgott, 2016)
- In the discrete case representation problems involving coverage (and uniformity) are particularly difficult.

L. Shao and M. Ehrgott. Discrete representation of non-dominated sets in multi-objective linear programming. *EJOR*, 2016. doi: http://dx.doi.org/10.1016/j.ejor.2016.05.001.

Hypervolume Problem

$Y_N = \{y_1, \ldots, y_n\} \subseteq \mathbb{R}^2$ and let $k \in \{1, \ldots, n\}$.
Find a subset $R \subseteq Y_N$, $|R| = k$, such that the area dominated by the points in R is maximized.

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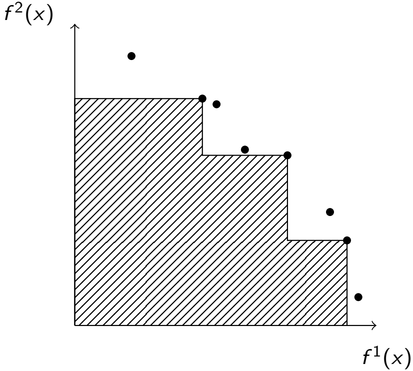
I. e. for a maximization problem (with $p \geq 0 \quad \forall y \in Y_N$):

$$\max_{x \in X} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

$$Vol\left(\bigcup_{y \in R} \left(y - \mathbb{R}^Q_+\right) \cap \mathbb{R}^Q_+\right)$$



E. Zitzler and L. Thiele. Multiobjective evolutionary algorithms: a comparative case study and the strength pareto approach. *IEEE Trans. Evol. Comput.*, 3(4):257–271, 1999.

Find one Point Maximizing the Hypervolume

We consider a integer linear biobjective maximization problem $(a, b \geq 0)$:

$$\max_{x \in X} \begin{pmatrix} a^\top x \\ b^\top x \end{pmatrix}$$

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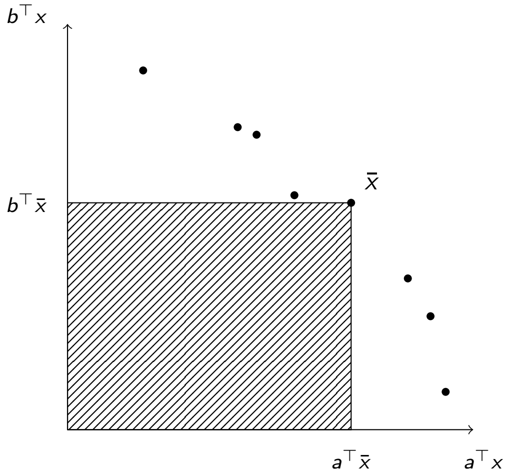

Find one Point Maximizing the Hypervolume

The solution $\bar{x} \in X$ maximizing the hypervolume can be computed by

$$\max_{x \in X} \; x^\top a b^\top x \tag{$*$}$$

- optimization of a quadratic objective function over the same constraint set

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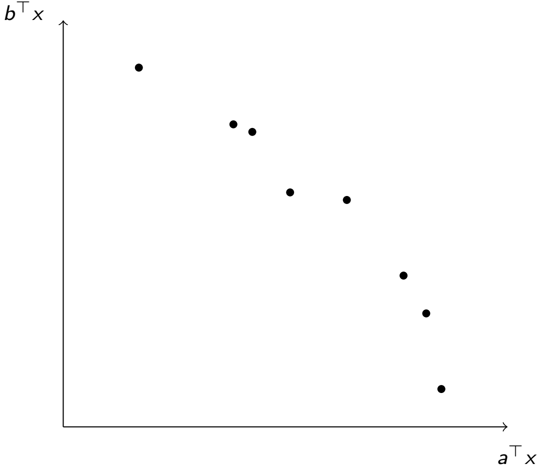
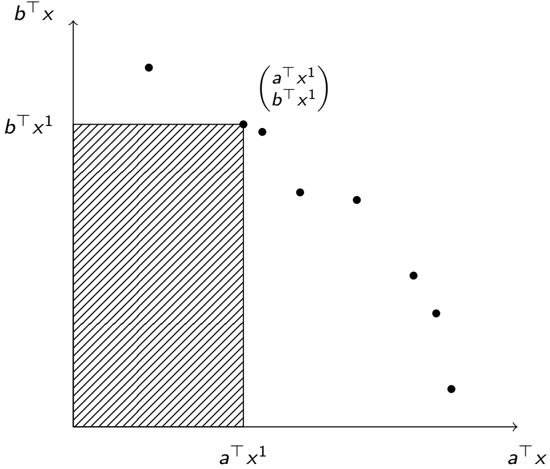
  - by shifting the reference point of the hypervolume every nondominated point can be generated by $(*)$

Computing an Optimal Representation wrt. Hypervolume – A Priori



Computing an Optimal Representation wrt. Hypervolume – A Priori



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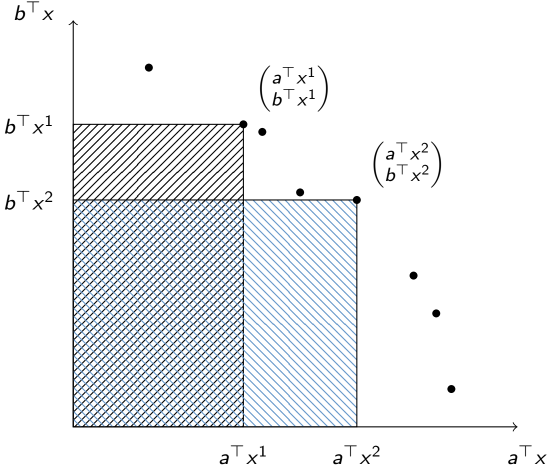

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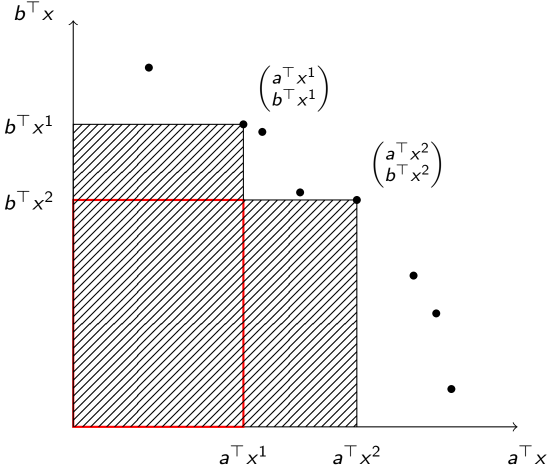

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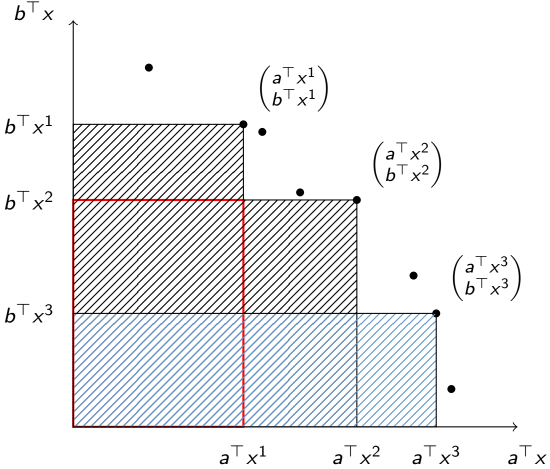

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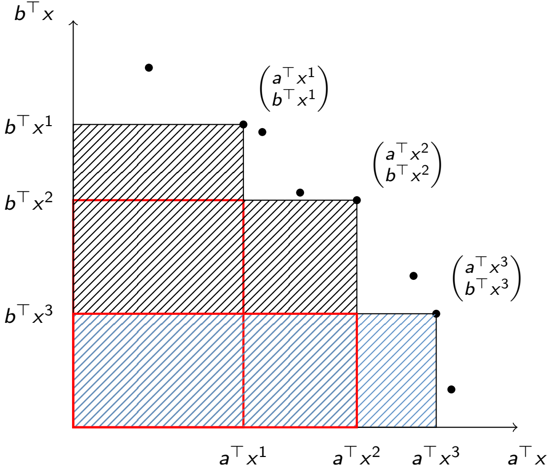

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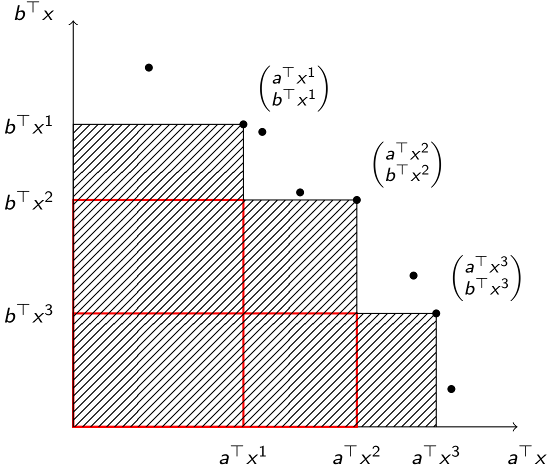
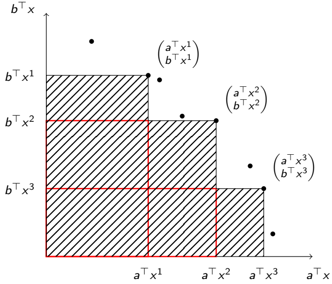


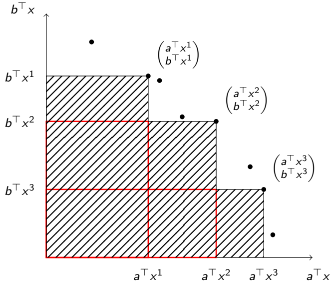
Computing an Optimal Representation wrt. Hypervolume – A Priori

$$V(x_1, x_2) = x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - x_1^\top ab^\top x_2$$



Computing an Optimal Representation wrt. Hypervolume – A Priori

$$
\begin{aligned}
V(x_1, x_2) &= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - {\color{red} x_1^\top ab^\top x_2} \\
V(x_1, x_2, x_3) &= V(x_1, x_2) + x_3^\top ab^\top x_3 - {\color{red} x_2^\top ab^\top x_3} \\
&= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 + x_3^\top ab^\top x_3 - {\color{red} x_1^\top ab^\top x_2} - {\color{red} x_2^\top ab^\top x_3}
\end{aligned}
$$



Computing an Optimal Representation wrt. Hypervolume – A Priori

$$
\begin{aligned}
V(x_1, x_2) &= x_1^\top a b^\top x_1 + x_2^\top a b^\top x_2 - x_1^\top a b^\top x_2 \\
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\end{aligned}
$$

$$
\begin{array}{rrcll}
\max & \multicolumn{3}{l}{\mathbf{x}^\top C\, \mathbf{x}} & \\
\text{s.t.} & a^\top x_i & \leq & a^\top x_{i+1} & \forall i \in \{1, \ldots, k-1\} \\
& x_i & \in & X & \forall i \in \{1, \ldots, k\} \\
& \mathbf{x} & = & (x_1^\top, x_2^\top, \ldots, x_k^\top)^\top &
\end{array}
$$

Computing an Optimal Representation wrt. Hypervolume – A Priori

$$\begin{aligned}
V(x_1, x_2) &= x_1^\top ab^\top x_1 + x_2^\top ab^\top x_2 - {\color{red} x_1^\top ab^\top x_2} \\
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\end{aligned}$$

$$\begin{array}{rrcll}
\max & \mathbf{x}^\top C\, \mathbf{x} & & & \\
\text{s.t.} & a^\top x_i & \leq & a^\top x_{i+1} & \forall i \in \{1, \ldots, k-1\} \\
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& \mathbf{x} & = & (x_1^\top, x_2^\top, \ldots, x_k^\top)^\top &
\end{array}$$

with

$$C = \begin{pmatrix} \boxed{ab^\top} & \boxed{-ab^\top} & & \\ & \boxed{ab^\top} & \boxed{-ab^\top} & \\ & & \ddots & \ddots \end{pmatrix}$$

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- A priori representation problems wrt. hypervolume can be formulated as quadratic optimization problems.
- Open topic: Optimal representations for discrete multiobjective optimization problems with $Q > 2$.

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