A coverage-based Box-Algorithm to compute a representation for optimization problems with three objective functions

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Recent Advances in
Multi-Objective Optimization
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Thanks to . . .

Bundesministerium
für Bildung
und Forschung

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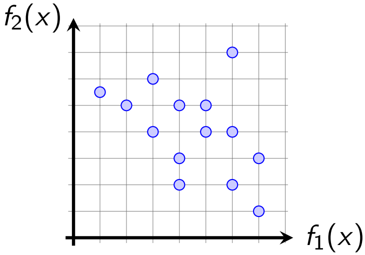
Multiple Objective Programming

Problem (Multiple Objective Programming Problem)

Let $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, p\} =: [p]$.

$$\text{(MOP)} \quad \min_{x \in X} f(x) := (f_1(x), \ldots, f_p(x))$$



Optimality Concept

Definition

$x^\star \in X$ efficient $:\Leftrightarrow \nexists x \in X : f(x) \leq f(x^\star)$

$:\Leftrightarrow \nexists x \in X \; \forall i \in [p] : f_i(x) \leq f_i(x^\star) \wedge f(x) \neq f(x^\star)$

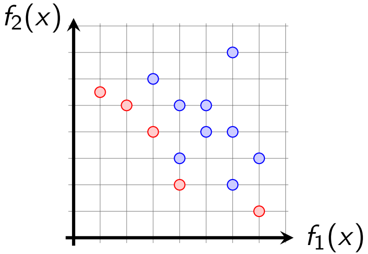
$y^\star \in Y = f(X)$ nondominated $:\Leftrightarrow y^\star = f(x^\star), x^\star$ efficient

X_E: Set of efficient solutions

Y_N: Set of nondominated points

y^I: Ideal point

y^N: Nadir point



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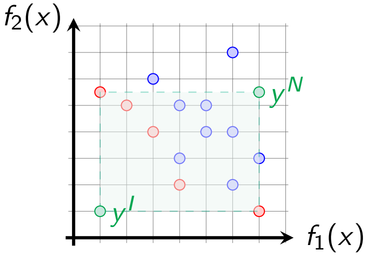

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General goal:
Compute the nondominated set Y_N (and present it to the decision maker)!

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- ... be *very large* in practice,
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$\rightsquigarrow$ compute *representation* of Y_N, i. e., an appropriate substitute for the nondominated set.

Representative System

Definition (Representative System)

For some MOP with outcome set Y, we call a finite approximation $Rep \subseteq Y$ representative system and its elements representative points.

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What is a "good" representative system?

Quality Measures

Definition (Sayin 2000, Ruzika 2007)

a) *Coverage error:* $\max_{y \in Y_N} \min_{z \in Rep} \|z - y\|$.

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c) *Cardinality:* $|Rep|$.

d) *Representation error:* $\max_{z \in Rep} \min_{y \in Y_N} \|y - z\|$.

Some Literature using Boxes for MOPs

Boxes / rectangles / cuboids . . . are frequently used in multiple objective programming:

- Laumans et al. (2006), Dhaenens et al. (2010), Kirlik and Sayin (2014): exact nondominated set by fixing one objective and projecting the others; grid-based structure; ε-constraint method.
- Dächert and Klamroth (2013): Improvement of splitting of boxes + generic algorithm; ε-constraint or Tchebycheff method
- Boland et al. (2014): Partition of projected search space by L-shapes and rectangles; 3-objective integer problems; experimental quality assessment.
- + several other approaches, e. g. in evolutionary algorithms.

Our Contribution

- **Goal:** Simple algorithm for computing a representative system
  - with desired coverage error
  - for MOPs with $p = 3$ objectives
  - with the capability of relating the run time of the algorithm to the quality of the representative system.

Our Contribution

- **Goal:** Simple algorithm for computing a representative system
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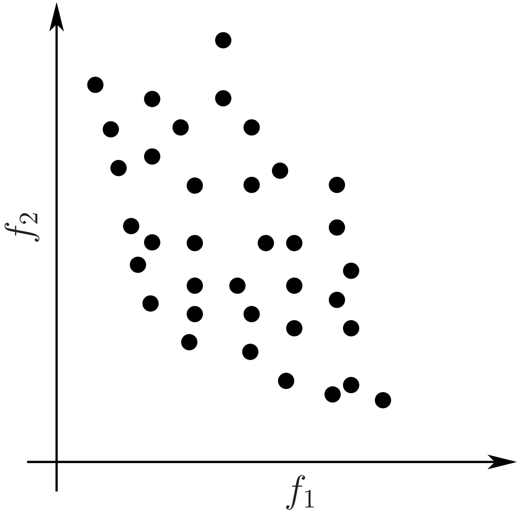
  - for MOPs with $p = 3$ objectives
  - with the capability of relating the run time of the algorithm to the quality of the representative system.
- **Idea:** Extending the Box-Algorithm of [Hamacher et al. 2007] to the case of three objective functions

Hamacher, Pedersen, Ruzika
Finding representative sytems for discrete bicriterion optimization problems
Operations Research Letters 35(3): 336-344, 2007

Short Review of the Box-Algorithm (2D)



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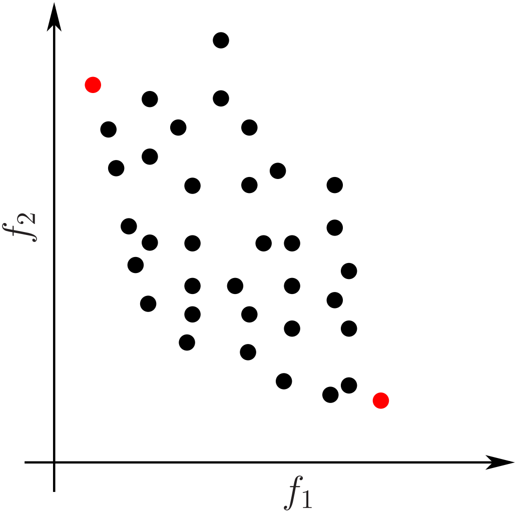

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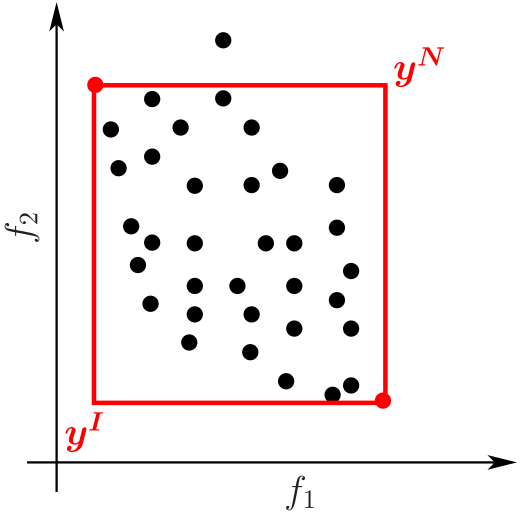

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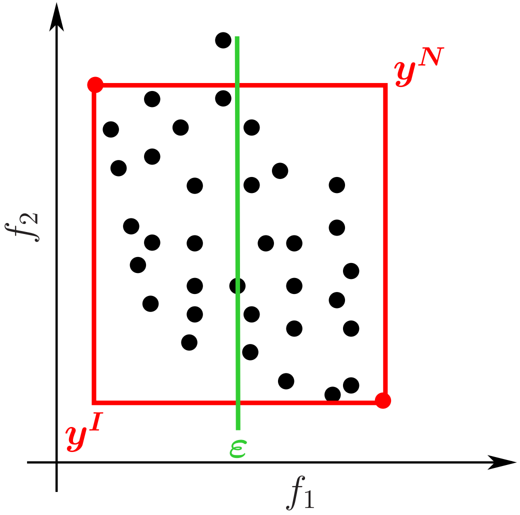

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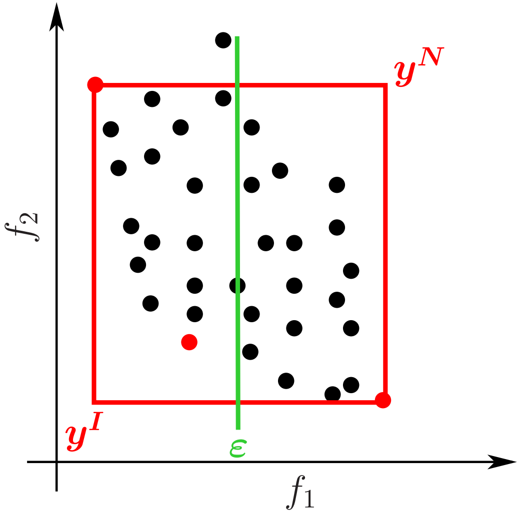

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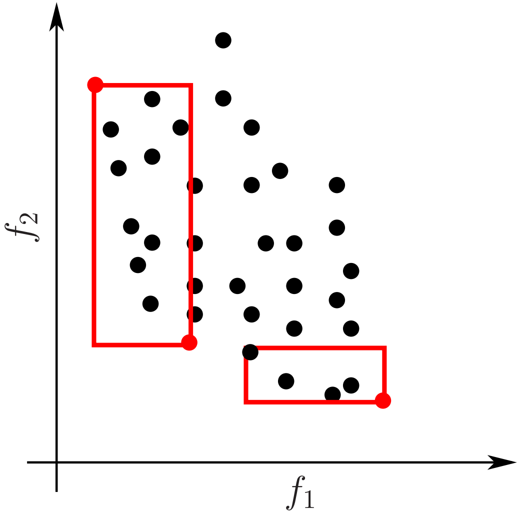

Box-Algorithm for MOPs with Three Objectives

Initialization

$$\text{(MOP)} \quad \min \; (f_1(x), f_2(x), f_3(x))$$
$$\text{s. t.} \quad x \in X \subseteq \mathbb{R}^n$$

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Definition

Let $\ell, u \in \mathbb{R}^3$ with $\ell \leqq u$. We refer to the cuboid

$$B(\ell, u) := \ell + \mathbb{R}^3_{\geqq} \cap u - \mathbb{R}^3_{\geqq} = \{y \in \mathbb{R}^3 | \ell \leqq y \leqq u\}$$

as the box defined by ℓ and u.

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Lemma

Let $\ell^0 \in y^I - \mathbb{R}^3_{\geqq}$ and let $u^0 \in y^N + \mathbb{R}^3_{\geqq}$. Then $Y_N \subseteq B(\ell^0, u^0)$.

Update Step

Definition

Let $\ell, u \in \mathbb{R}^3, \ell \leqq u$ and let $\varepsilon_i = \ell_i + \frac{u_i - \ell_i}{2}$ for $i = 1, 2, 3$. Then, we define the lexicographic ε-constraint scalarizations with lower bounds $(P^1_{\varepsilon_1,\varepsilon_2}), (P^2_{\varepsilon_1,\varepsilon_3})$, and $(P^3_{\varepsilon_2,\varepsilon_3})$ as

$$
\begin{aligned}
(P^1_{\varepsilon_1,\varepsilon_2}) \qquad & \operatorname{lex\,min} \ \ (f_3(x),\ f_2(x),\ f_1(x)) \\
& \text{s.t.} \ \ x \in X \\
& \left.\begin{array}{l} \ell_1 \leq f_1(x) \leq \varepsilon_1 \\ \ell_2 \leq f_2(x) \leq \varepsilon_2 \\ \ell_3 \leq f_3(x)\,(\leq u_3) \end{array}\right\} =: f(x) \in B(\ell, u)^{(\varepsilon_1,\varepsilon_2,u_3)}
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and, analogously,

$$(P^2_{\varepsilon_1,\varepsilon_3}) \ \operatorname{lex\,min} \ (f_2(x),\ f_1(x),\ f_3(x)) \quad \text{s.t.} \ x \in X, \quad f(x) \in B(\ell, u)^{(\varepsilon_1,u_2,\varepsilon_3)}$$

$$(P^3_{\varepsilon_2,\varepsilon_3}) \ \operatorname{lex\,min} \ (f_1(x),\ f_3(x),\ f_2(x)) \quad \text{s.t.} \ x \in X, \quad f(x) \in B(\ell, u)^{(u_1,\varepsilon_2,\varepsilon_3)}.$$

Update Step

Proposition

Let z^* be the image under f of an optimal solution of $(P^1_{\varepsilon_1,\varepsilon_2})$. Then, there does not exist a $y \in Y_N \setminus \{z^*\}$ such that

$$y \in B(z^*, u) \cup B\left(\ell, (\varepsilon_1, \varepsilon_2, z_3^*)^\top\right) \setminus B\left((\ell_1, z_2^*, z_3^*)^\top, (z_1^*, \varepsilon_2, z_3^*)^\top\right)$$

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Proof.

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Proof.

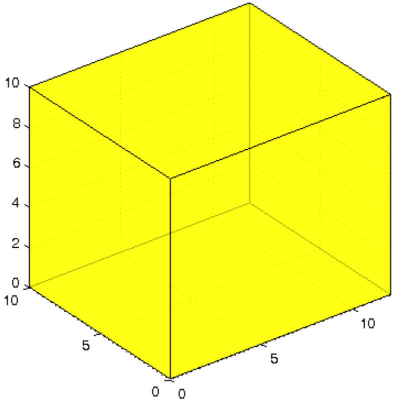
- $B(z^*, u)$ is dominated by z^*
- A nondominated point in $B\left(\ell, (\varepsilon_1, \varepsilon_2, z_3^*)^\top\right) \setminus B\left((\ell_1, z_2^*, z_3^*)^\top, (z_1^*, \varepsilon_2, z_3^*)^\top\right)$ contradicts optimality of z^* for $(P^1_{\varepsilon_1,\varepsilon_2})$

□

Update Step

Example:

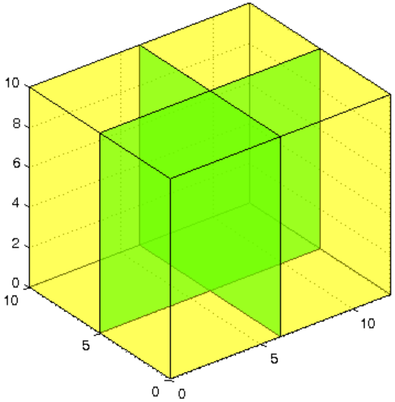
Consider an initial box with $\ell = 0$ and $u = (12,\ 10,\ 10)^\top$.



Update Step

Example:

Solve $(P^1_{\varepsilon_1,\varepsilon_2})$ with $\boldsymbol{\varepsilon_1 = 6}$ **and** $\boldsymbol{\varepsilon_2 = 5}$.



Update Step

Example:

Optimal solution with image $z^* = (2,\ 3,\ 4)^\top$

Update Step

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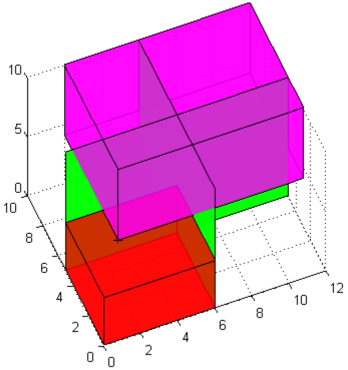
Optimal solution with image $z^* = (2,\ 3,\ 4)^\top \rightsquigarrow$ Cut-off Regions (see first proposition): $B\left((2,3,4)^\top,(12,10,10)^\top\right)$ und $B\left(0,(6,5,4)^\top\right) \setminus B\left((0,3,4)^\top,(2,5,4)^\top\right)$



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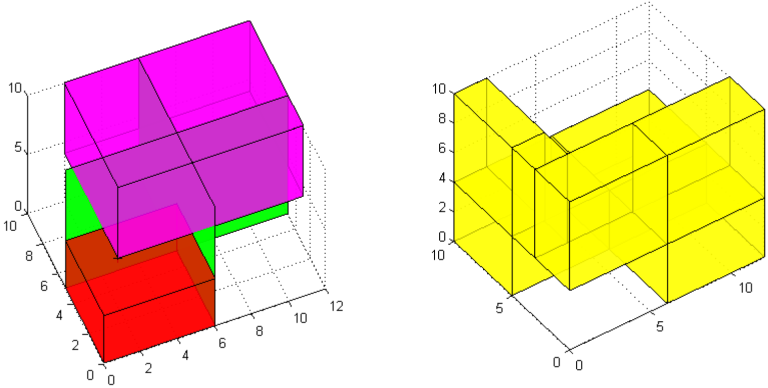

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Let $\ell, u \in \mathbb{R}^3, \ell \leqq u$ and let $\varepsilon_i = \ell_i + \frac{u_i - \ell_i}{2}$ for $i = 1, 2, 3$. Suppose $(P^1_{\varepsilon_1, \varepsilon_2})$ is solved. Then, the four quarters of the current box $B(\ell, u)$ are defined as

$$Q_{1,1} := B\left(\ell, \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ u_3 \end{pmatrix}\right), \quad Q_{1,2} := B\left(\begin{pmatrix} \varepsilon_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}, \begin{pmatrix} u_1 \\ \varepsilon_2 \\ u_3 \end{pmatrix}\right),$$

$$Q_{1,3} := B\left(\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \ell_3 \end{pmatrix}, u\right), \quad Q_{1,4} := B\left(\begin{pmatrix} \ell_1 \\ \varepsilon_2 \\ \ell_3 \end{pmatrix}, \begin{pmatrix} \varepsilon_1 \\ u_2 \\ u_3 \end{pmatrix}\right).$$

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$$B^{1,1} := B\left(\begin{pmatrix}\ell_1\\ z_2^*\\ z_3^*\end{pmatrix}, \begin{pmatrix}z_1^*\\ \varepsilon_2\\ u_3\end{pmatrix}\right), \quad B^{1,2} := B\left(\begin{pmatrix}\ell_1\\ \ell_2\\ z_3^*\end{pmatrix}, \begin{pmatrix}\varepsilon_1\\ z_2^*\\ u_3\end{pmatrix}\right)$$

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$$B^{1,5} := B\left(\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \ell_3\end{pmatrix}, \begin{pmatrix}u_1\\ u_2\\ z_3^*\end{pmatrix}\right), \quad B^{1,6} := B\left(\begin{pmatrix}\ell_1\\ \varepsilon_2\\ \ell_3\end{pmatrix}, \begin{pmatrix}\varepsilon_1\\ u_2\\ z_3^*\end{pmatrix}\right)$$

$$B^{1,7} := B\left(\begin{pmatrix}\ell_1\\ \varepsilon_2\\ z_3^*\end{pmatrix}, \begin{pmatrix}z_1^*\\ u_2\\ u_3\end{pmatrix}\right)$$

Properties of the Subdivision

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$$Y_N \cap B(\ell, u) \subseteq \bigcup_{i=1}^{7} B^{1,i}$$

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Observation

$$Vol(B^{1,i}) \leq \frac{1}{4} \cdot Vol(B(\ell, u))$$

Moreover, for each of the quarters $Q_{1,1}$, $Q_{1,2}$ and $Q_{1,3}$, we can find two pairs of boxes for which the combined volume fulfills this formula.

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$B^{\text{lex}} := B\left(\ell, (\varepsilon_1, \varepsilon_2, z_3^*)^\top\right)$ and $B^{\text{dom}} := B\left(z^*, u\right)$ are cut off with

$$Vol(B^{\text{lex}}) + Vol(B^{\text{dom}})$$

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=\ & Vol\left(B\left((\varepsilon_1, \varepsilon_2, \ell_3)^\top, (u_1, u_2, z_3^*)^\top\right)\right) + Vol\left(B\left((\varepsilon_1, \varepsilon_2, z_3^*)^\top, u\right)\right) \\
=\ & Vol\left(B\left((\varepsilon_1, \varepsilon_2, \ell_3)^\top, u\right)\right) = Vol(Q_{1,3}) = \frac{1}{4} Vol(B(\ell, u))
\end{aligned}$$

□

Correctness

Algorithm 1 Box-Algorithm for three objectives

Require: MOP with three objectives, $\delta^C > 0$
Ensure: *Rep* representative system with coverage error at most δ^C

$\mathcal{S} := \{\text{InitialBox}()\}$
while $\mathcal{S} \neq \emptyset$ **do**
  $B := B(\ell, u) := \text{SelectBox}(\mathcal{S})$
  if $\|\ell - u\|_\infty \leq \delta^C$ **then**
    Use $(P^1_{u_1,u_2})$ to search for a representative point in B
  else
    Determine the 2 longest edges of B
    Solve $(P^j_{\cdot,\cdot})$, $j \in \{1, 2, 3\}$, dividing these 2 edges
    Add optimal outcome z^* to *Rep*
    for $i \in \{1, \dots, 7\}$ **do**
      Add $B^{j,i}$ to $\mathcal{S}$

Correctness

Algorithm 1 Box-Algorithm for three objectives

Require: MOP with three objectives, $\delta^C > 0$
Ensure: *Rep* representative system with coverage error at most δ^C

```
S := {InitialBox()}
while S ≠ ∅ do
    B := B(ℓ, u) := SelectBox(S)
    if ‖ℓ − u‖∞ ≤ δ^C then
        Use (P^1_{u1,u2}) to search for a representative point in B
    else
        Determine the 2 longest edges of B
        Solve (P^j_{·,·}), j ∈ {1, 2, 3}, dividing these 2 edges
        Add optimal outcome z* to Rep
        for i ∈ {1, ..., 7} do
            Add B^{j,i} to S
```

Correctness

Theorem

Algorithm 1 terminates in finitely many steps. It outputs a collection of boxes containing all nondominated points. The representative system Rep has a coverage error of at most δ^C (w. r. t. $\|\cdot\|_\infty$). More precisely, the algorithm performs at most $\mathcal{O}\left(\left(\frac{L}{\delta^C}\right)^{2\cdot\log_2(7)}\right)$ many iterations, where L is the distance of the corner points of the initial box $B(\ell^0, u^0)$, i.e., $L := \|\ell^0 - u^0\|_\infty$.

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Proof.

see our paper . . . □

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Corollary

Let Rep be a representative system with coverage error less than or equal to δ^C *(w. r. t.* $\|\cdot\|_\infty$*) and let* Rep_N *denote all points of Rep which are not dominated by any other point in this set. Then it is*

$$Y_N \subseteq \left(Rep_N - (\delta^C, \delta^C, \delta^C)^\top\right) + \mathbb{R}^3_{\geqq}.$$

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- Coverage property: $\exists z \in Rep$ with $\|y - z\|_\infty \leq \delta^C$ which implies

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- If $z \notin Rep_N$, then there exists $\hat{z} \in Rep_N$ with $\hat{z} \leq z$.

□

Selection Rule

max-dist selection rule

Corollary

Let $\textsc{SelectBox}()$ *always select the box with largest corner point distance. Suppose the algorithm is aborted prematurely after* $\Gamma \geq 1$ *iterations and for all remaining boxes* $B \in \mathcal{S}$*, we additionally execute a "completion step". Then, the representative system Rep has a coverage error of at most* $L \cdot 2^{-\lfloor (\log_7(6\Gamma+1))/2 \rfloor}$*, where L equals the corner point distance* $\|\ell^0 - u^0\|_\infty$ *of the initial box* $B(\ell^0, u^0)$*.*

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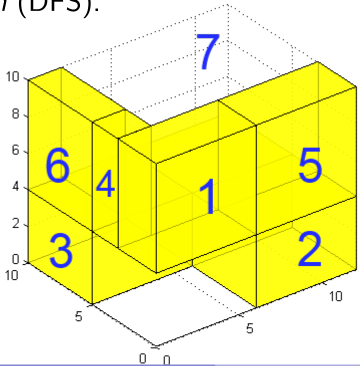

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- Store the *list of unexplored boxes* in analogy to a tree, where we perform a *depth first search* (DFS).
- *Result:* By appropriately cutting off dominated parts of (partial) dominated boxes during the algorithm, we obtain a representative system $Rep = Rep_N$ fulfilling the desired accuracy.

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Definition

Let $\mathcal{B}$ be the last subdivision (covering the whole set Y_N) obtained from Algorithm 1. Let $z \in Rep$ and $B(z) \subseteq \mathcal{B}$ be the set containing either the box for which z was computed or the corresponding child boxes contained in the corresponding quarter of the box for which z was computed. Then, we define $\mathcal{B}^z := \left\{B \in \mathcal{B} : B \cap \left(z - \mathbb{R}^3_{\geq}\right) \neq \emptyset \wedge B \notin B(z)\right\}$. If $\mathcal{B}^z \neq \emptyset$, we call z a *critical representative point*.

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Lemma

For MOP, let $(Rep, \mathcal{B})$ be the output of Algorithm 1 and $z \in Rep$ be some representative point. Then, it holds

$$(\mathcal{B}^z = \emptyset) \Longrightarrow (z \in Y_N)$$

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where $B^{Rep}_{\delta^C}(z - \mathbb{R}^3_{\geq}) := \left\{ y \in Rep : \exists \tilde{y} \in z - \mathbb{R}^3_{\geq} \wedge \|y - \tilde{y}\| \leq \delta^C \right\}$ *and* $\max_{B(\ell,u) \in \mathcal{B}^z} \|\ell - z\|$ *returns the value* 0 *if* $\mathcal{B}^z = \emptyset$.

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Conclusion

Box algorithm for computing representative sets for MOPs with $p = 3$ objectives

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Thank you for your attention!