

Robust and Multiobjective Optimisation: Opportunities and Challenges

Recent Advances in Multi-Objective Optimization

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Lancaster University

Management School

Robust Optimisation

Multi-Objective Optimisation

- optimise with several objectives



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In this talk:

What can MOO and RO learn from each other?

Structure

We discuss three connections:



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- 1 adding robustness as a new objective function to a single-objective problem

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- 2 considering the robust counterpart of a multi-objective problem



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We discuss three connections:

- 1 adding robustness as a new objective function to a single-objective problem
- 2 considering the robust counterpart of a multi-objective problem
- 3 using a multi-objective perspective on a robust problem

Adding robustness as a new objective function



Idea

$$\begin{aligned} \min f(x, \xi) \\ x \in \mathcal{X}(\xi) \end{aligned}$$



Adding Robustness

Idea

$$\min_{x \in \mathcal{X}(\xi)} f(x, \xi)$$

→

$$\begin{aligned} \min \tilde{f}(x) \\ \max \text{rob}(x) \\ x \in \mathcal{X} \end{aligned}$$



Examples

- Mulvey, Vanderbei, Zenios. *Robust Optimization of Large-Scale Systems*, Oper Res '95.
- Liebchen, Lübbecke, Möhring, Stiller. *The Concept of Recoverable Robustness, Linear Programming Recovery, and Railway Applications*, LNCS 5868, '09.
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Approach of [CGS16]

$$\begin{aligned} \min f(x, \xi) \\ F(x, \xi) \leq 0 \\ x \in \mathcal{X} \end{aligned}$$



Approach of [CGS16]

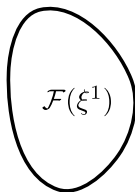
$$\begin{array}{l} \min f(x, \xi) \\ F(x, \xi) \leq 0 \\ x \in \mathcal{X} \end{array} \quad \rightarrow \quad \min \left(\begin{array}{l} \\ F(y(\xi), \xi) \leq 0 \quad \forall \xi \in \mathcal{U} \\ x \in \mathcal{X} \\ y : \mathcal{U} \rightarrow \mathcal{X} \end{array} \right)$$

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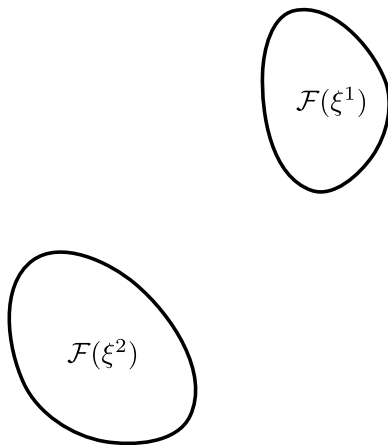
Adding Robustness

ε -constraint on f



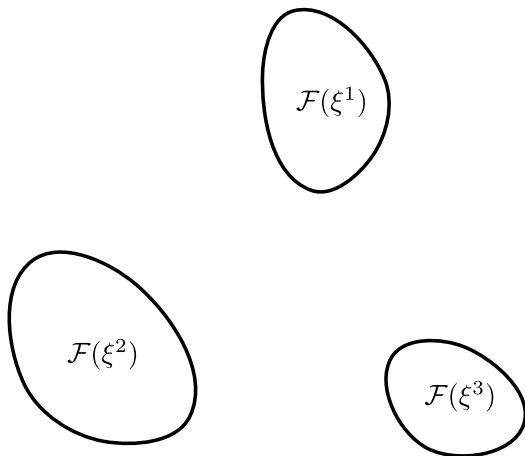
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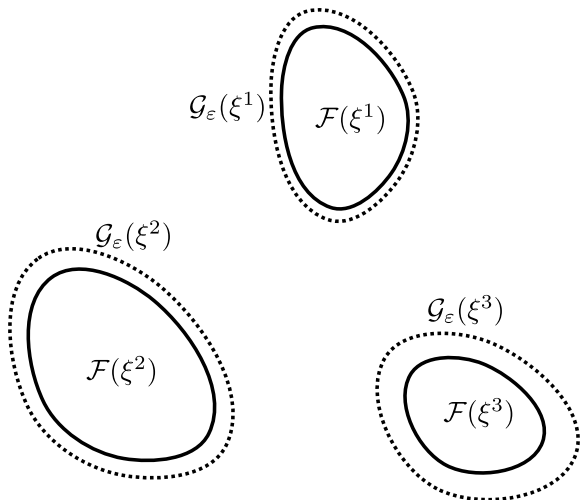
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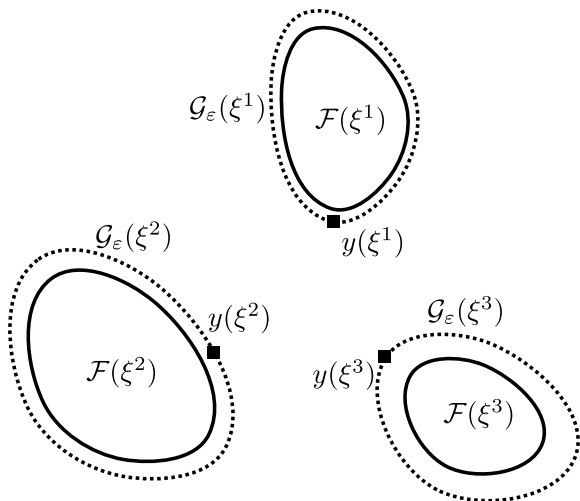
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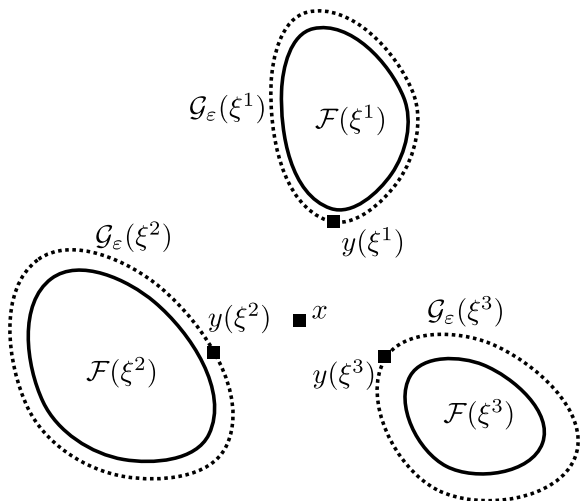
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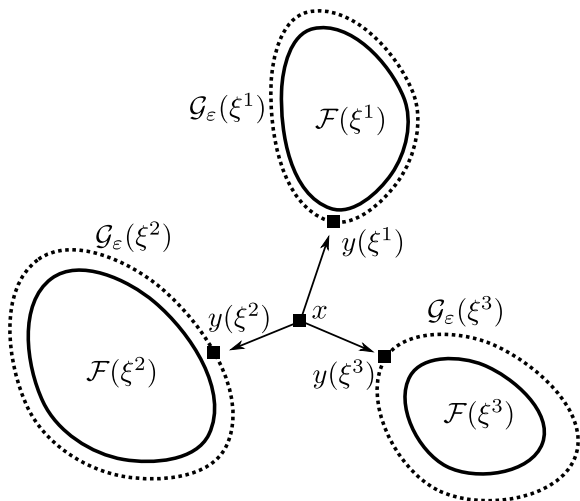
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Results on Biobjective Problem

Let

- $\mathcal{U} = \text{conv}(\mathcal{U}')$ with $\mathcal{U}' := \{\xi^1, \dots, \xi^N\}$.
- F consist of m constraints with $F_i : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$, $i = 1, \dots, m$
- $f : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$ jointly quasiconvex in (y, ξ)
- $d(x, \cdot)$ quasiconvex.
- \mathcal{X} convex

Then $\text{Rec}(\mathcal{U})$ and $\text{Rec}(\mathcal{U}')$ have the same set of recoverable-robust solutions.

Approach Summary

- widely used
- most often:



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- most often:
 - sum of objectives, one arbitrary scaling factor
 - one arbitrary budget on (robustness) objective



Approach Summary

- widely used
- most often:
 - sum of objectives, one arbitrary scaling factor
 - one arbitrary budget on (robustness) objective
- no effort to find (all/most) Pareto solutions
- multi-objective nature not acknowledged

Robust counterparts of multi-objective problems

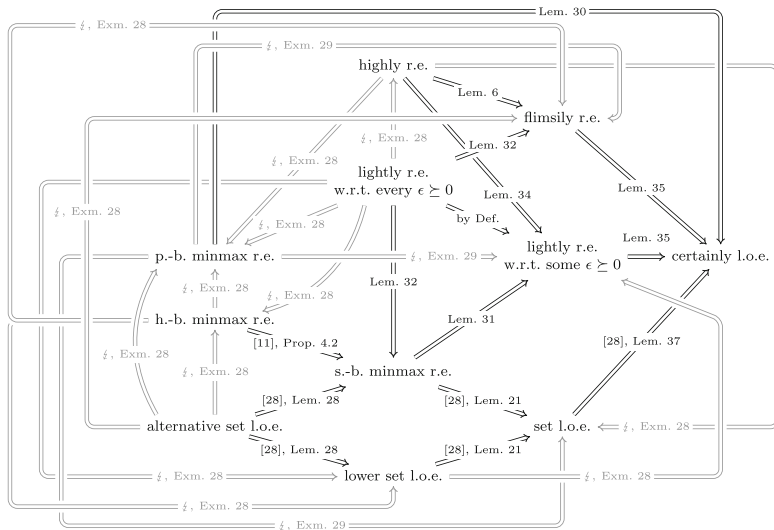


Examples

- recently developed
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- Kuhn, Raith, Schmidt, Schöbel. *Bi-objective robust optimisation*, EJOR '16.
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Approach Summary

- young and increasingly popular
- see Gabi's talk
- see Corinna's talk



Using a multi-objective perspective on robust problems



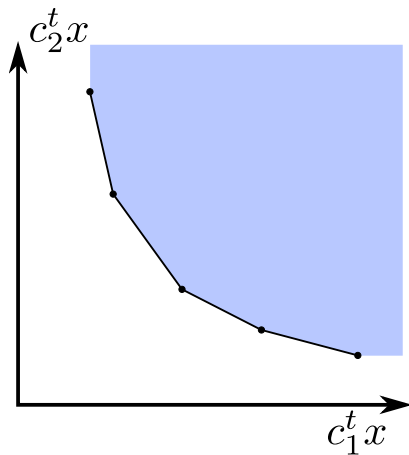
Examples

- Aissi, Bazgan, Vanderpooten. *Approximation of min–max and min–max regret versions of some combinatorial optimization problems*, EJOR '07.
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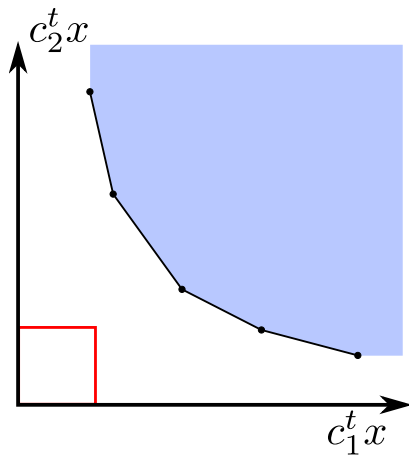
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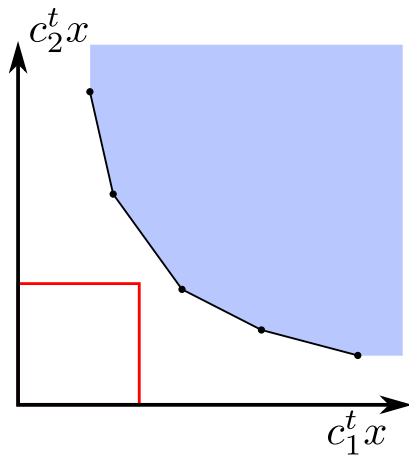
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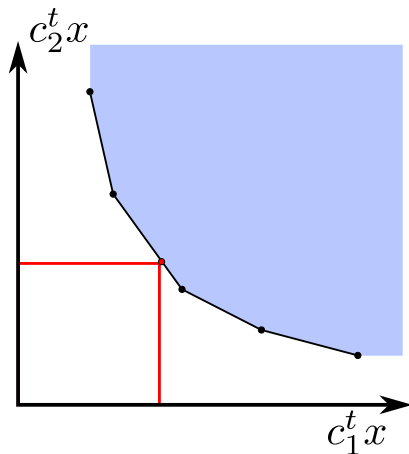
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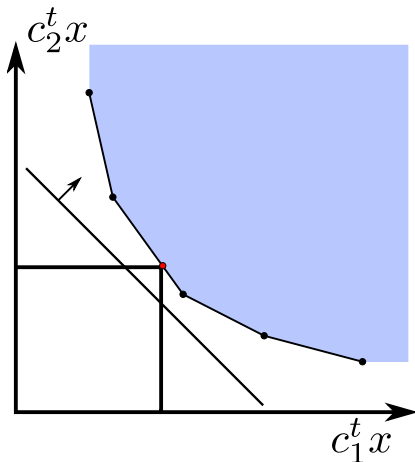
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How to use this relationship?

For approximation algorithms:

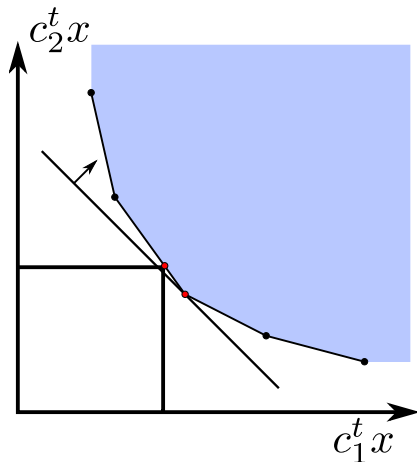
- average solution
 - $\min c_1^t x + c_2^t x$



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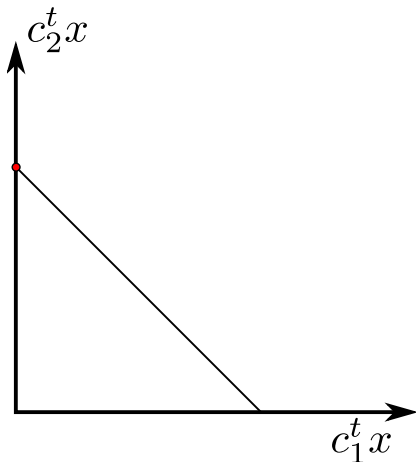
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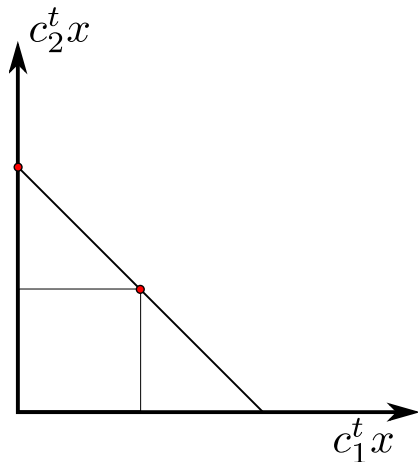
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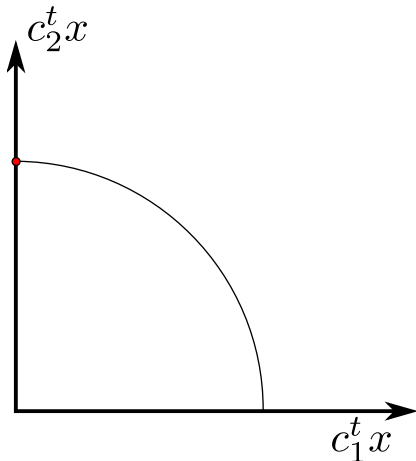
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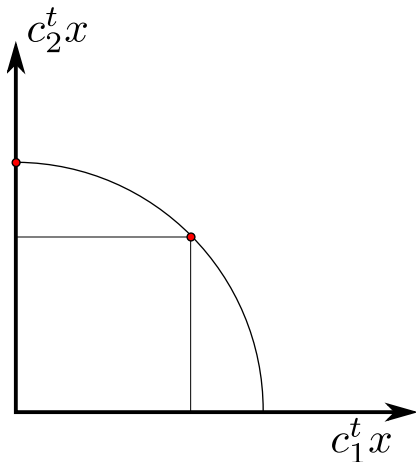
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 - \sqrt{N} approx



Computing the AC-WC Curve

Approach to Use Relationship

Consider again our first connection between robust and multi-objective optimisation, using robustness as an additional objective:

$$\min_{x \in \mathcal{X}} \left(\begin{array}{c} \hat{c}x \\ \max_{k \in [N]} c^k x \end{array} \right)$$

Computing the AC-WC Curve

Observations

- As for the WC-solution, we can show:
 - The set of Pareto efficient solution w.r.t. $(c_1, \dots, c_N, \hat{c})$ contains the Pareto efficient solutions w.r.t. (z, \hat{c})



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- Solving weighted sums w.r.t. $(c_1, \dots, c_N, \hat{c})$ is **easy** (algorithms of original problem type can be used), but w.r.t. (z, \hat{c}) is **hard** (algorithms for robust problem must be used)



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Idea

Solve “the right” subproblems w.r.t. $(c_1, \dots, c_N, \hat{c})$ instead of (z, \hat{c}) !



Computing the AC-WC Curve

Master problem

Let $\{y^1, \dots, y^r\}$ denote the extreme points of all Pareto efficient solutions w.r.t. $(c_1, \dots, c_N, \hat{c})$ in the **objective** space.

Computing the AC-WC Curve

Master problem

Let $\{y^1, \dots, y^r\}$ denote the extreme points of all Pareto efficient solutions w.r.t. $(c_1, \dots, c_N, \hat{c})$ in the **objective** space. We solve weighted sums

$$(M) \quad \min \lambda a + (1 - \lambda)z$$

$$\text{s.t. } \hat{y} = \sum_{i=1}^r \alpha_i y^i$$

$$\sum_{i=1}^r \alpha_i = 1$$

$$\hat{y}_{n+1} = a$$

$$\hat{y}_i \leq z$$

$$i = 1, \dots, N$$

$$\alpha \geq 0$$

Computing the AC-WC Curve

Slave problems

Problem (M) potentially contains exponentially many variables.



Computing the AC-WC Curve

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Use **column generation** to generate them.



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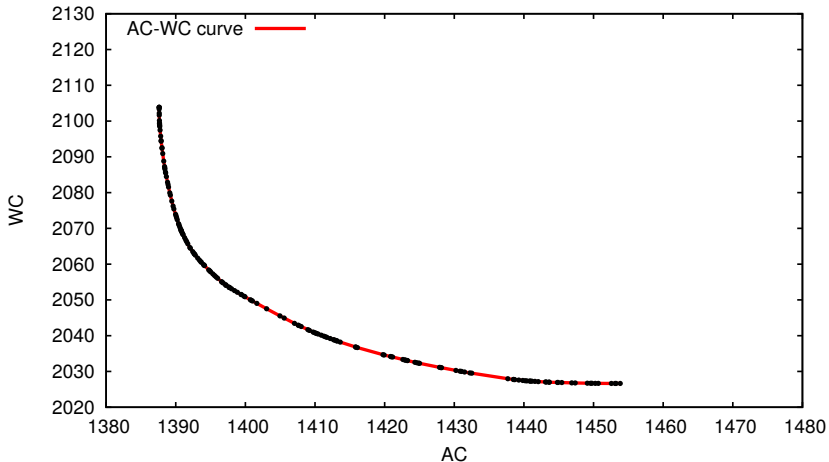
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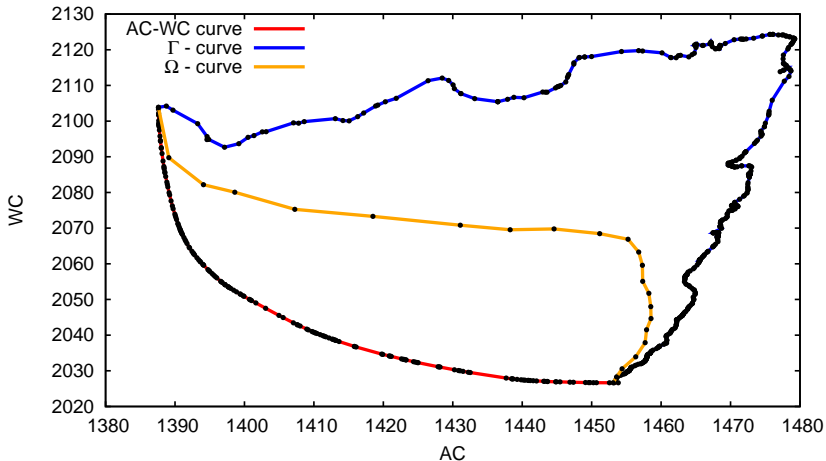
Result

We can compute the AC-WC Pareto front by solving problems of type (M) and problems of the original type.

Computing the AC-WC Curve



Computing the AC-WC Curve



Variable-Sized Uncertainty

Idea



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$$\min_{x \in \mathcal{X}} \max_{c \in \mathcal{U}_\lambda} c^t x$$

with $\mathcal{U} = \hat{c} + \lambda B$



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with $\mathcal{U} = \hat{c} + \lambda B$

- We know the shape of uncertainty.
- We do not know the size of uncertainty.
- Find smallest set \mathcal{S} that contains an optimal robust solution for every λ .



Connection to MOO

- Let $B = \prod_{i \in [n]} [-d_i, d_i]$.

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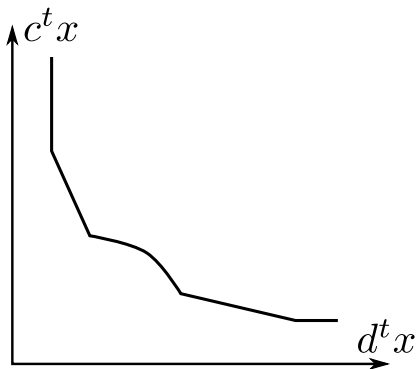
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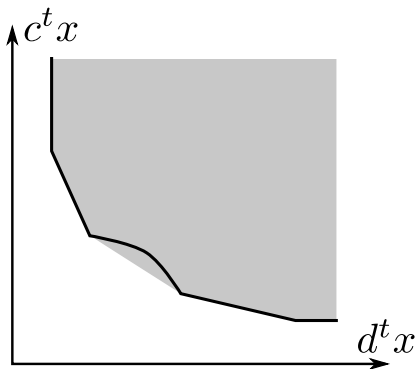


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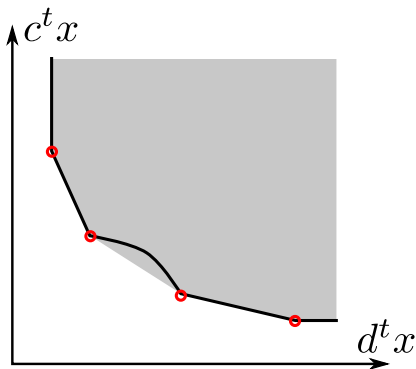


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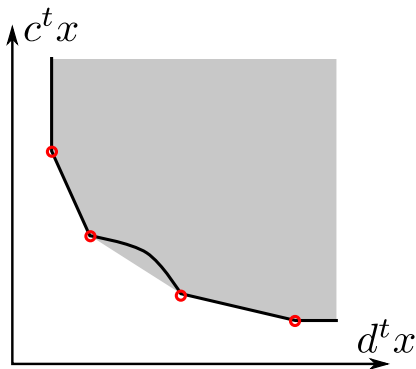
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Connection to MOO

- Let $B = \prod_{i \in [n]} [-d_i, d_i]$.

$$\max_{c \in \mathcal{U}_\lambda} c^t x = c^t x + \lambda d^t x$$

- Find minimal set of efficient extreme points!



Variable-Sized Uncertainty

Application to Shortest Path

- Compute K efficient points in $O(K)$



Variable-Sized Uncertainty

Application to Shortest Path

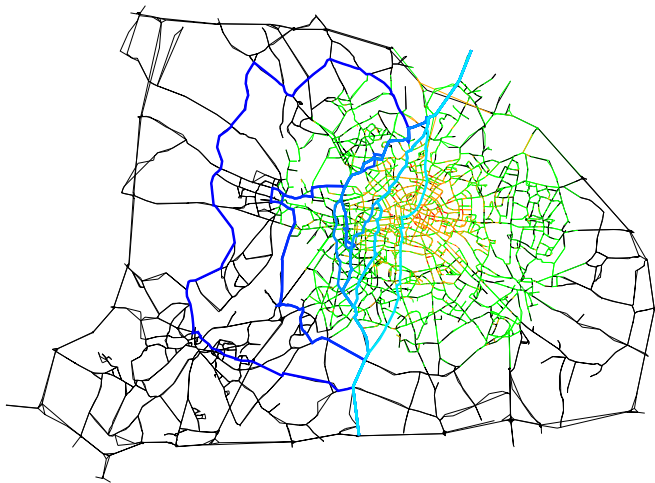
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- General: $K \in 2^{\Omega(\log^2(n))}$



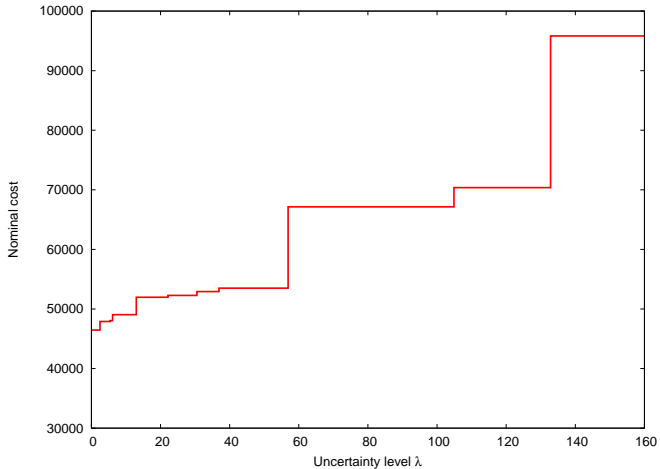
Application to Shortest Path

- Compute K efficient points in $O(K)$
- General: $K \in 2^{\Omega(\log^2(n))}$
- Acyclic: $K \in O(n^{\log n})$
- Layered graph: $K \in O(w^{\log(\ell+1)})$
- Series-parallel: $K \in O(m - n)$

Variable-Sized Uncertainty



Variable-Sized Uncertainty



Robust and MO Optimisation

- strengthen RO via MOO
- strengthen MOO via RO

Robust and MO Optimisation

- strengthen RO via MOO
- strengthen MOO via RO
- Plenty of combinations possible!
- Many more exciting combinations to come!