Unconstrained Binary Multiobjective Optimization: Weight Space Decomposition, Arrangements of Hyperplanes and Zonotopes

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"Multiobjective Combinatorial Optimization: Beyond the Biobjective Case"

Introduction

Unconstrained Binary Multiobjective Optimization Problem

$$\begin{aligned} \text{vmax} \quad & f(x) = \left(\sum_{i=1}^{n} p_i^1 x_i, \sum_{i=1}^{n} p_i^2 x_i, \ldots, \sum_{i=1}^{n} p_i^m x_i \right) \quad \text{(mo.0c)} \\ \text{s.t.} \quad & x_i \in \{0,1\} \quad \forall i \in \{1,\ldots,n\}. \end{aligned}$$

with $p_i = (p_i^1, \ldots, p_i^m)^\top \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, $\forall i \in \{1, \ldots, n\}$

Notation

- $\mathcal{X} = \{0,1\}^m$ set of feasible solutions

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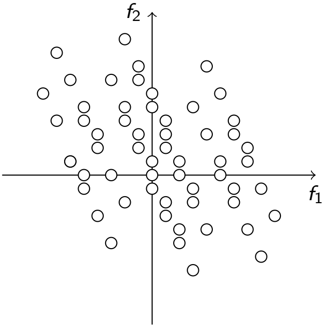

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- $\mathcal{Y} = f(\mathcal{X})$ set of feasible points

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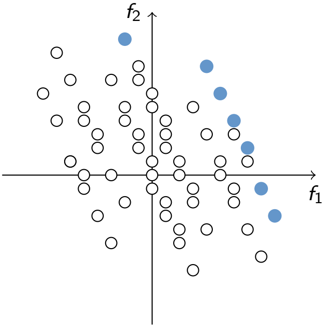

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- $\mathcal{Y} = f(\mathcal{X})$ set of feasible points
- $\mathcal{Y}_N = \mathcal{Y}_{sN} \cup \mathcal{Y}_{uN} \subset \mathcal{Y}$ set of (supported/unsupported) nondominated points

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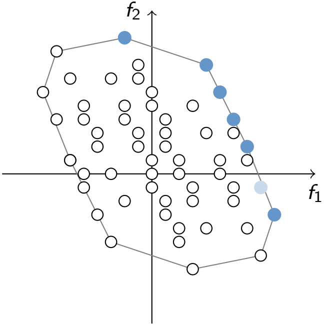

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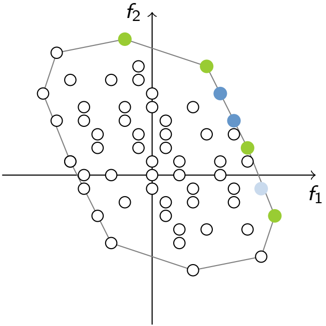

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- $\mathcal{Y}_{eN}$ set of extreme supported nondominated points

Goal

Computation of extreme supported solutions:

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Weighted sum scalarization

$$\max \quad \sum_{j=1}^{m} \lambda_j \cdot \sum_{i=1}^{n} p_i^j x_i$$
$$\text{s.t.} \quad x_i \in \{0,1\} \quad \forall i \in \{1,\dots,n\}.$$

with $\lambda \in \widetilde{\mathcal{W}}^0 = \left\{(\lambda_1,\dots,\lambda_m) \in \mathbb{R}^m_> : \sum_{j=1}^{m} \lambda_j = 1\right\}$.

Arthur M. Geoffrion
Proper Eficiency and the Theory of Vector Maximization
Journal of Mathematical Analysis and Applications, 1968

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Computation of extreme supported solutions:

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with $\lambda \in \mathcal{W}^0 = \left\{ (\lambda_2, \ldots, \lambda_m) \in \mathbb{R}_{>}^{m-1} \; : \; \sum_{j=2}^{m} \lambda_j < 1 \right\}$ and $\lambda_1 = \left(1 - \sum_{j=2}^{m} \lambda_j \right)$.

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Weight Space Decomposition

Let $\bar{y} \in \mathcal{Y}_{sN}$.

$$\mathcal{W}^0(\bar{y}) = \Big\{ (\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 \,:\, \lambda_1 = (1 - \sum_{j=2}^{m} \lambda_j), \\ \sum_{j=1}^{m} \lambda_j \bar{y}_j \geq \sum_{j=1}^{m} \lambda_j y_j, \forall y \in \mathcal{Y} \Big\}$$

Anthony Przybylski, Xavier Gandibleux, and Matthias Ehrgott
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INFORMS Journal on Computing, 2010

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Weight Space Decomposition

Let $\bar{y} \in \mathcal{Y}_{sN}$.

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- $\mathcal{W}^0(\bar{y})$ is a convex polytope
- $\bar{y}$ is extreme supported $\Leftrightarrow$ $\mathcal{W}^0(\bar{y})$ has dimension $m-1$

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- $\mathcal{W}^0(\bar{y})$ is a convex polytope
- $\bar{y}$ is extreme supported $\Leftrightarrow$ $\mathcal{W}^0(\bar{y})$ has dimension $m-1$
- $\mathcal{W}^0 = \bigcup_{\bar{y} \in \mathcal{Y}_{eN}} \mathcal{W}^0(\bar{y})$

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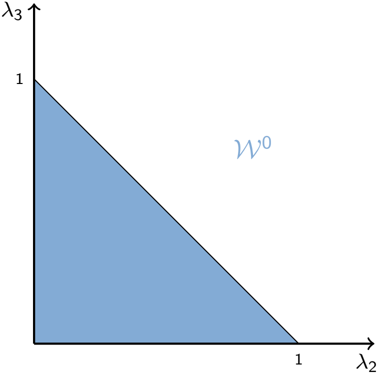

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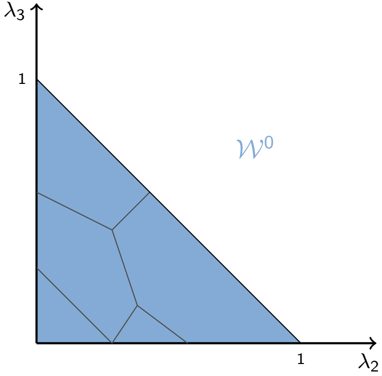

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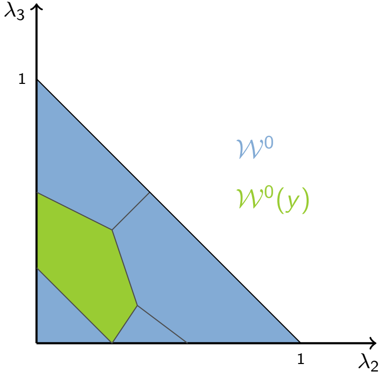

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Definitions

Closed line segment

Let u and v be vectors in $\mathbb{R}^m$. The closed line segment $[u, v]$ is defined as

$$[u, v] = \{y \in \mathbb{R}^m : y = u + \mu \cdot (v - u), \mu \in [0, 1]\}.$$

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Minkovski sum

Let $\mathcal{A}$, $\mathcal{B} \subset \mathbb{R}^m$. The Minkowski sum of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} + \mathcal{B} = \{a + b \,:\, a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Definitions

Zonotope

A zonotope $\mathcal{Z}$ is defined as the Minkovski sum of a finite number of closed line segments $[u_i, v_i]$ with vectors u_i and $v_i \in \mathbb{R}^m$, $i \in \{1, ..., n\}$.

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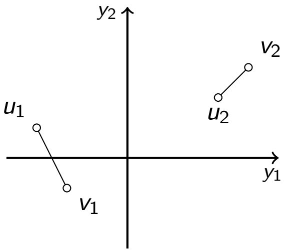

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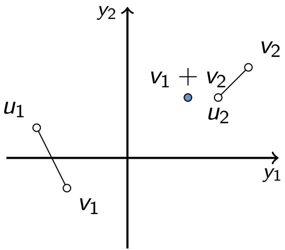

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B. Schulze,
M. Stiglmayr,
K. Klamroth

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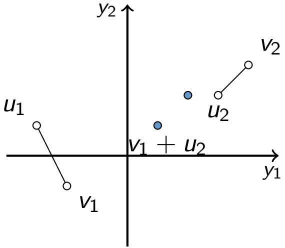

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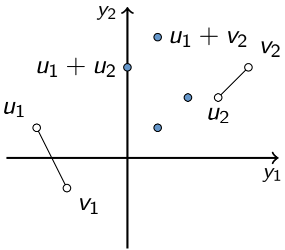

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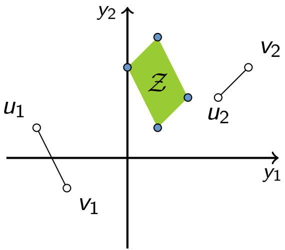

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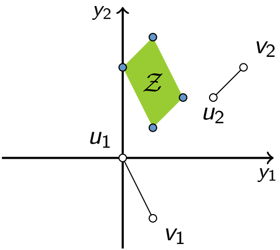

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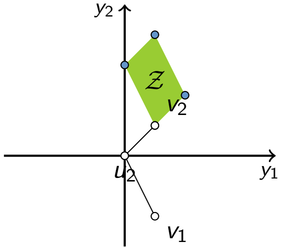

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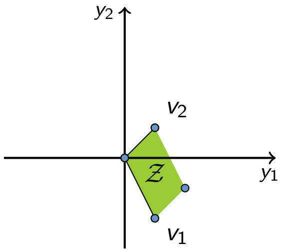

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Hyperplane

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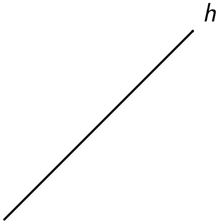

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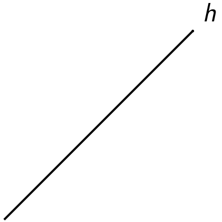
A hyperplane $h \subset \mathbb{R}^m$ is defined as the affine hull of m affinely independent points in $\mathbb{R}^m$.

A hyperplane h subdivides $\mathbb{R}^m$ into two open half spaces, denoted as h^+ and h^-.



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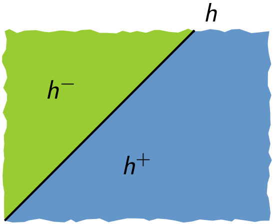
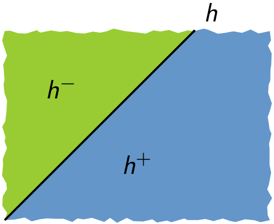


Definitions

Arrangement of hyperplanes

Given a finite set of hyperplanes h_i in $\mathbb{R}^m$, the hyperplanes subdivide $\mathbb{R}^m$ into connected polytopes of different dimensions. This is called the arrangement of hyperplanes.



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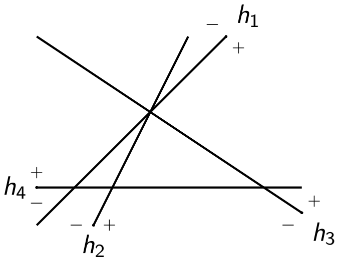
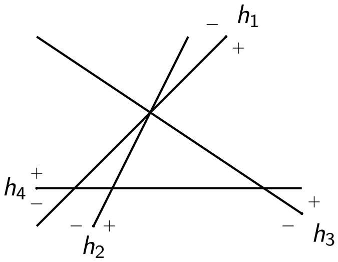


Definitions

Position vector

Let $y \in \mathbb{R}^m$. The position vector $\mathrm{P}(y) = (\mathrm{P}_1(y), ..., \mathrm{P}_n(y))$ of y is defined by

$$\mathrm{P}_i(y) = \begin{cases} -1 & \text{if } y \in h_i^- \\ 0 & \text{if } y \in h_i \\ +1 & \text{if } y \in h_i^+ \end{cases} \quad \forall i \in \{1, ..., n\}.$$



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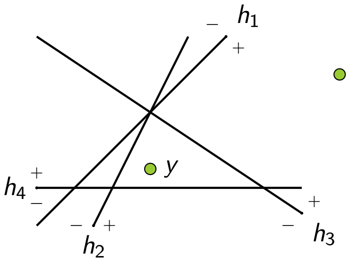

$\mathrm{P}(y) = (+1, +1, -1, +1)$

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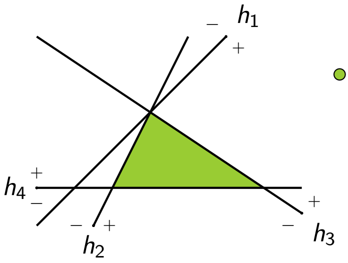

- m-face or cell with $\mathrm{P}(y) = (+1, +1, -1, +1)$

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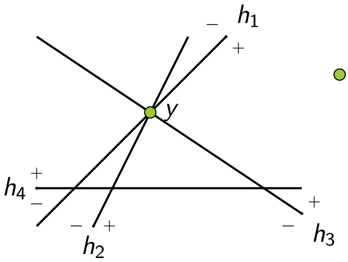
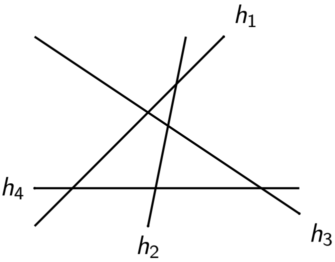


- k-face ($k = 0$) with $\mathrm{P}(y) = (0, 0, 0, +1)$

Definitions

Simple arrangement of hyperplanes

Hyperplanes $h_1, ..., h_n$ in $\mathbb{R}^m$, $m \leq n$

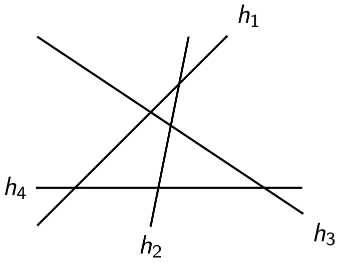


Definitions

Simple arrangement of hyperplanes

Hyperplanes $h_1, ..., h_n$ in $\mathbb{R}^m$, $m \leq n$

- Intersection of any subset of m hyperplanes is a unique point

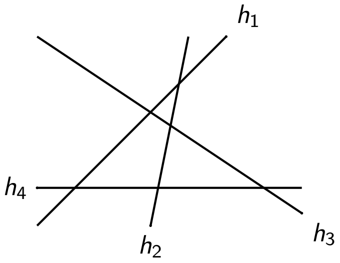


Definitions

Simple arrangement of hyperplanes

Hyperplanes $h_1, ..., h_n$ in $\mathbb{R}^m$, $m \leq n$

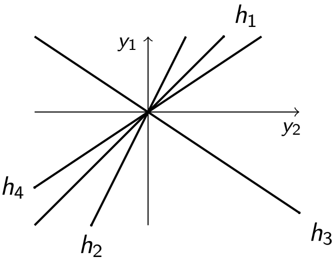
- Intersection of any subset of m hyperplanes is a unique point
- Intersection of any subset of $(m+1)$ hyperplanes is empty



Definitions

Central arrangement of hyperplanes

Hyperplanes $h_1, ..., h_n$ in $\mathbb{R}^m$, $m \leq n$

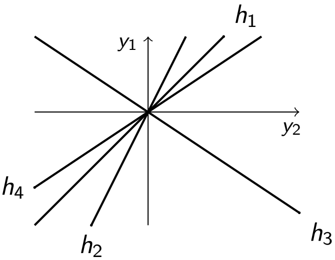


Definitions

Central arrangement of hyperplanes

Hyperplanes $h_1, \ldots, h_n$ in $\mathbb{R}^m$, $m \leq n$

- $(0, \ldots, 0)^\top$ is contained in every hyperplane



Interrelations

(mo.0c)

Interrelations

(mo.0c)

↕

Weight space
decomposition

Interrelations

(mo.0c)

↕

Weight space decomposition

Zonotope

Interrelations

(mo.0c) ↕ Weight space decomposition

Zonotope

Arrangement of hyperplanes

Interrelations

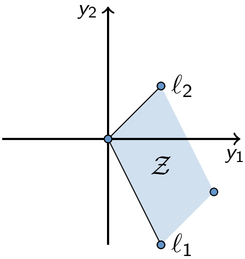
(mo.0c) ↕ Weight space decomposition

Zonotope ↕ Arrangement of hyperplanes

Arrangements and Zonotopes

Zonotope:

$$\ell_i = [0, p_i],\ p_i \in \mathbb{R}^m,\ i \in \{1, \ldots, n\}$$



Herbert Edelsbrunner
Algorithms in Combinatorial Geometry
Springer, 1987

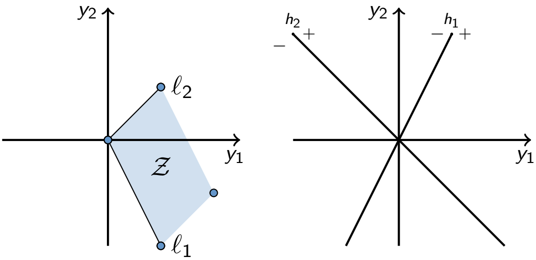
Arrangements and Zonotopes

Zonotope:

$$\ell_i = [0, p_i],\ p_i \in \mathbb{R}^m,\ i \in \{1, ..., n\}$$

Associated arrangement of hyperplanes:

$$h_i = \{y \in \mathbb{R}^m : \langle p_i, y\rangle = 0\} \text{ with}$$
$$h_i^+ = \{y \in \mathbb{R}^m : \langle p_i, y\rangle > 0\}$$
$$h_i^- = \{y \in \mathbb{R}^m : \langle p_i, y\rangle < 0\}$$



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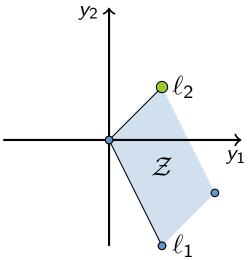
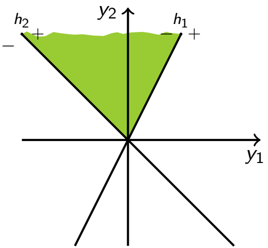
Arrangements and Zonotopes

This duality has an order reversing characteristic: A k-face of a zonotope in $\mathbb{R}^m$ corresponds to an $(m-k)$-face of the associated arrangement of hyperplanes.





Herbert Edelsbrunner
Algorithms in Combinatorial Geometry
Springer, 1987

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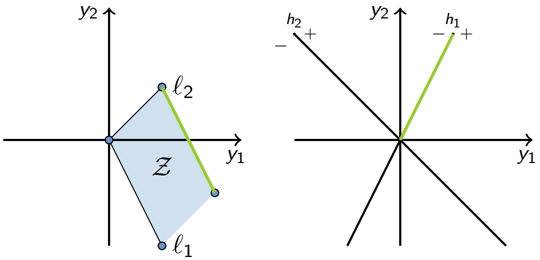

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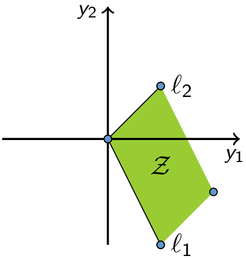

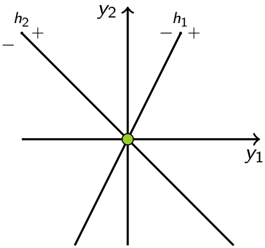

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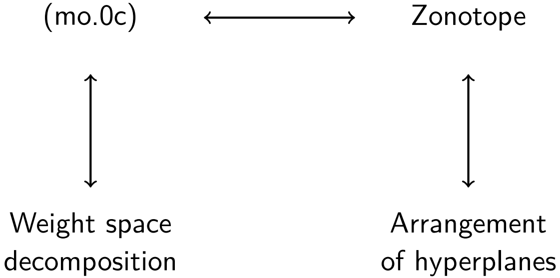
Interrelations

(mo.0c) $\longleftrightarrow$ Weight space decomposition

Zonotope $\longleftrightarrow$ Arrangement of hyperplanes

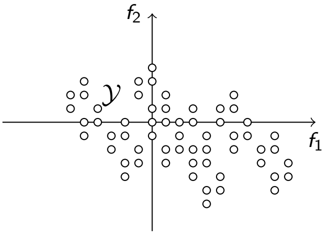
Interrelations



(mo.0c) and Zonotopes

Instance of (mo.0c):

$$p_i = (p_i^1, p_i^2, \dots, p_i^m)^\top,\ i \in \{1, \dots, n\}.$$



(mo.0c) and Zonotopes

Instance of (mo.0c):

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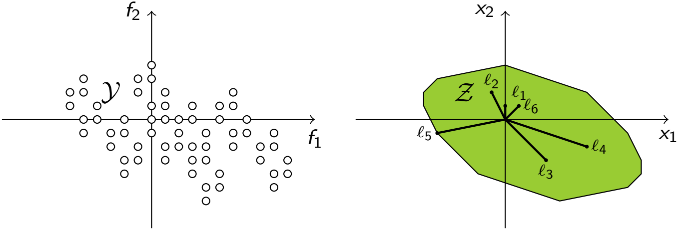
Associated zonotope:

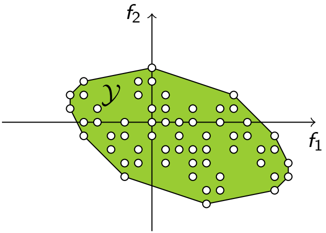
$$\ell_i = [0, p_i] \qquad i \in \{1, \dots, n\}$$

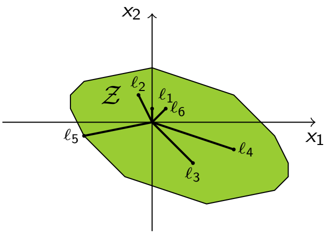


(mo.0c) and Zonotopes

It holds:

$$\mathcal{Z} = \text{conv}(\mathcal{Y})$$





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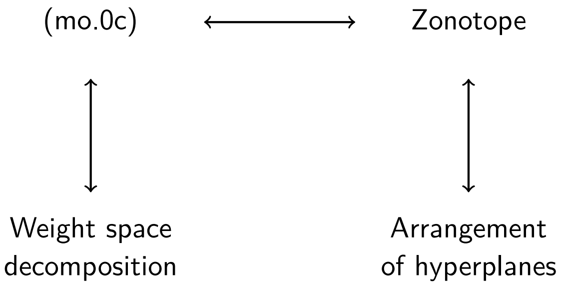

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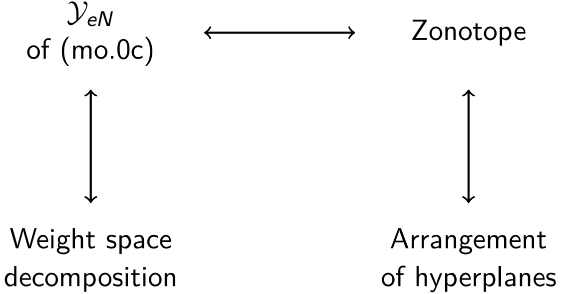


B. Schulze,
M. Stiglmayr,
K. Klamroth

Introduction
Definitions
Interrelations
Multiobjective knapsack problem
Summary

Interrelations

$\mathcal{Y}_{eN}$ of (mo.0c)	set of nondominated extreme points of Zonotope
$\updownarrow$	$\updownarrow$
Weight space decomposition	Arrangement of hyperplanes

Interrelations

$\mathcal{Y}_{eN}$ of (mo.0c) = set of nondominated extreme points of Zonotope

$\updownarrow$ Weight space decomposition

$\updownarrow$ Arrangement of hyperplanes

Interrelations

$\mathcal{Y}_{eN}$ of (mo.0c) $=$ set of nondominated extreme points of Zonotope

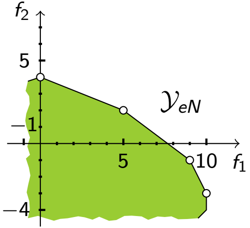
$\updownarrow$ $\qquad\qquad$ $\updownarrow$

Weight space decomposition $\qquad$ Intersection of Arrangement of hyperplanes with $\mathbb{R}^m_>$

Interrelations

$\mathcal{Y}_{eN}$ of (mo.0c)	$=$	set of nondominated extreme points of Zonotope
$\updownarrow$		$\updownarrow$
Weight space decomposition	$=$	Intersection of Arrangement of hyperplanes with $\widetilde{\mathcal{W}}^0 \subset \mathbb{R}^m_>$

(mo.0c), Arrangements and Weight Space



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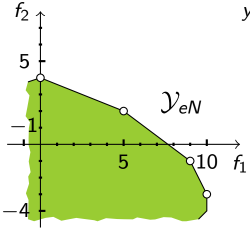

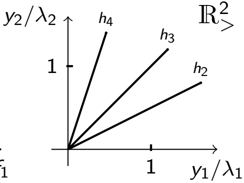

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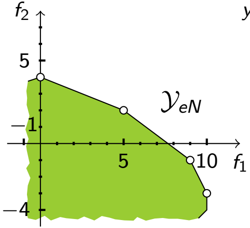
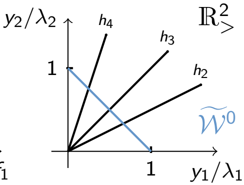
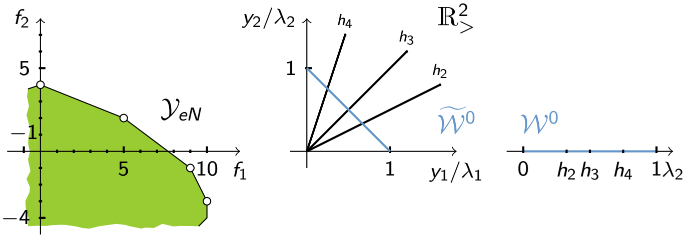




(mo.0c), Arrangements and Weight Space



Complexity

The number of cells in a central arrangement of hyperplanes in $\mathbb{R}^m$, m fixed, is bounded by

$$2 \cdot \sum_{i=0}^{m-1} \binom{n-1}{i}.$$

Thomas Zaslavsky
Facing up to arrangements: face-count formulas for partitions of space by hyperplanes
Memoirs of the American Mathematical Society, 1975

Complexity

The number of cells in a central arrangement of hyperplanes in $\mathbb{R}^m$, m fixed, is bounded by

$$2 \cdot \sum_{i=0}^{m-1} \binom{n-1}{i}.$$

- The same bound holds for the number of extreme supported solutions for (mo.0c).

Weight Space Decomposition

Each variable x_i defines a hyperplane

$$h_i = \left\{(\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 \,:\, \left(1 - \sum_{j=2}^{m} \lambda_j\right) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j = 0\right\}$$

Weight Space Decomposition

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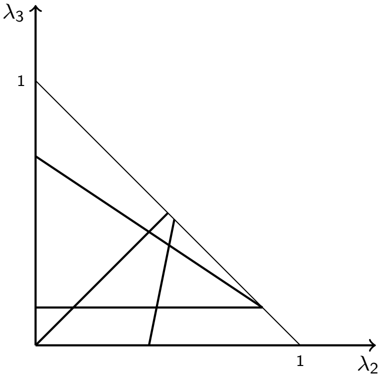
and half spaces

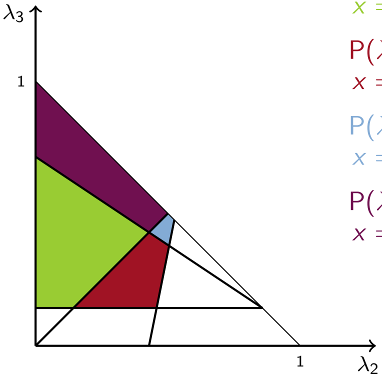
$$h_i^- = \left\{(\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 \;:\; \Big(1 - \sum_{j=2}^{m} \lambda_j\Big) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j < 0\right\} \quad (x_i = 0)$$

$$h_i^+ = \left\{(\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 \;:\; \Big(1 - \sum_{j=2}^{m} \lambda_j\Big) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j > 0\right\} \quad (x_i = 1)$$

Weight Space Decomposition



Weight Space Decomposition



$P(\lambda) = (-1, -1, -1, +1)^\top$
$x = (0, 0, 0, 1)^\top$

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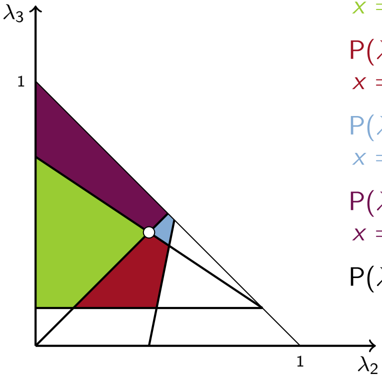

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Solution Approach

Input: hyperplanes h_i, $i \in \{1, ..., n\}$

for all intersection points λ of $(m-1)$ hyperplanes **do**
    if $\lambda \in \mathcal{W}^0$ **then**
        generate all solutions corresponding to λ

Output: set of supported solutions

Solution Approach

Input: hyperplanes h_i, $i \in \{1, ..., n\}$

```
for all intersection points λ of (m − 1) hyperplanes do
    if λ ∈ W^0 then
        generate all solutions corresponding to λ
```

Output: set of supported solutions

- For simple arrangements: Complexity $\mathcal{O}(n^m)$.

Solution Approach

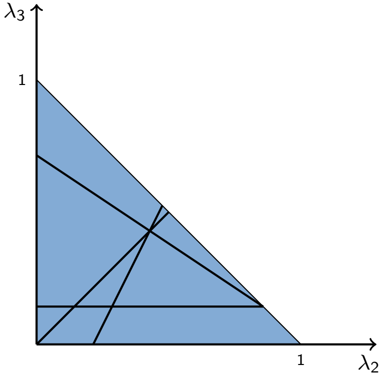
Input: hyperplanes h_i, $i \in \{1, ..., n\}$
1: **for** all intersection points λ of $(m-1)$ hyperplanes **do**
2: **if** $\lambda \in \mathcal{W}^0$ **then**
3: generate all solutions corresponding to λ
Output: set of supported solutions

- For simple arrangements: Complexity $\mathcal{O}(n^m)$.
- For nonsimple arrangements: Complexity $\mathcal{O}(2^n)$.

Nonextreme Supported Solutions

Nonextreme supported soultions:

occur if more than $m-1$ hyperplanes intersect in one point.



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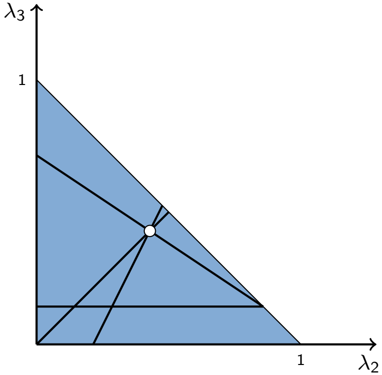

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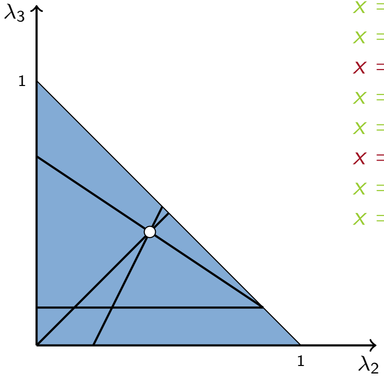

$P(\lambda) = (0,0,0,+1)^\top$

$x = (0,0,0,1)^\top$

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$x = (1,1,1,1)^\top$

Multiobjective Knapsack Problem (mo.1c)

$$\begin{aligned} \text{vmax} \quad & f(x) = \left(\sum_{i=1}^{n} p_i^1 x_i, \sum_{i=1}^{n} p_i^2 x_i, \dots, \sum_{i=1}^{n} p_i^m x_i \right) \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq W \\ & x_i \in \{0, 1\} \quad \forall i \in \{1, \dots, n\}. \end{aligned}$$

with $p_i = (p_i^1, \dots, p_i^m)^\top \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $0 < w_i < W$, $\forall i \in \{1, \dots, n\},\ W < \sum_{i=1}^{n} w_i$.

Solution Approach

Generalized versions of the dichotomic search.

Fritz Bökler and Petra Mutzel
Output-Sensitive Algorithms for Enumerating the Extreme Nondominated Points of Multiobjective Combinatorial Optimization Problems
Springer, 2015

Anthony Przybylski, Xavier Gandibleux, and Matthias Ehrgott
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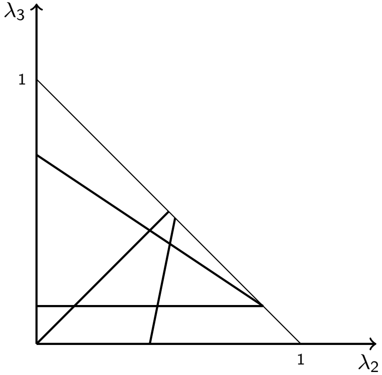
Weight Space Decomposition



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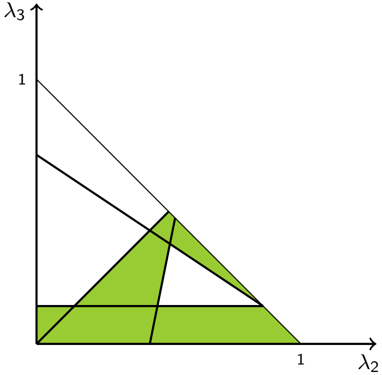

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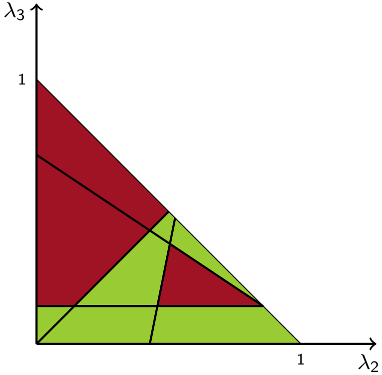
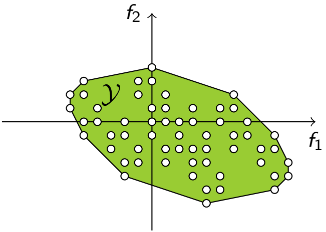


Summary

Summary



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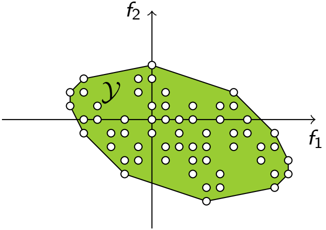

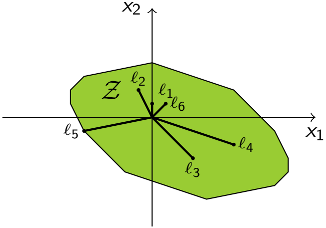

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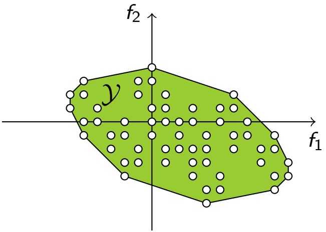
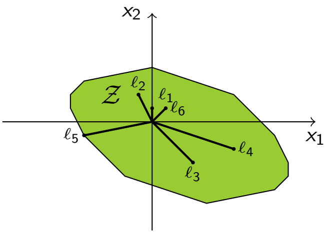
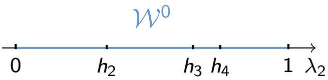


f_2
f_1
$\mathcal{Y}$

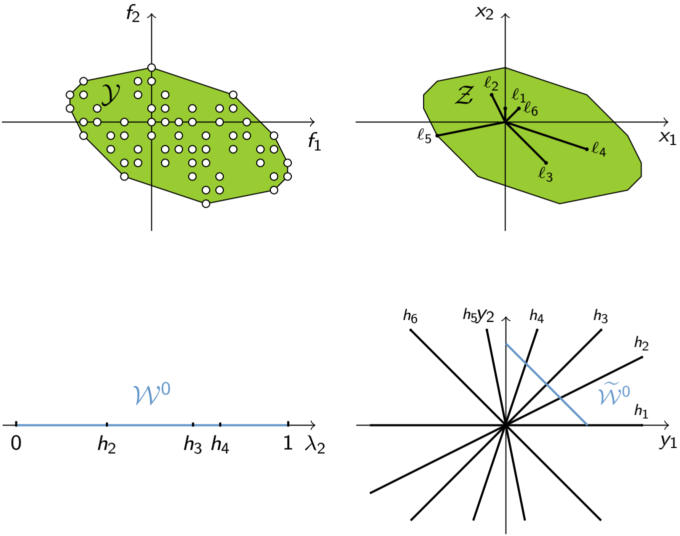


x_2
x_1
$\mathcal{Z}$
ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5
ℓ_6



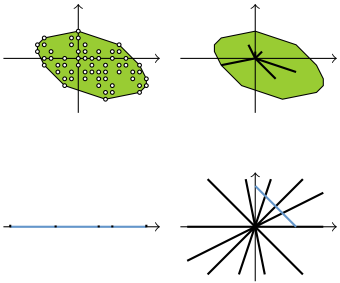
$\mathcal{W}^0$
0
h_2
h_3
h_4
1
λ_2

Summary



f_2
f_1
$\mathcal{Y}$
x_2
x_1
$\mathcal{Z}$
ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5
ℓ_6
$\mathcal{W}^0$
0
h_2
h_3
h_4
1
λ_2
h_6
h_5
y_2
h_4
h_3
h_2
h_1
$\widetilde{\mathcal{W}}^0$
y_1

Summary



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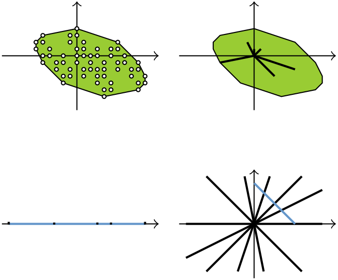

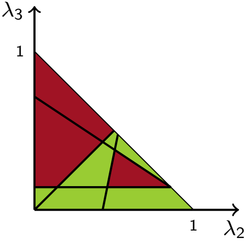

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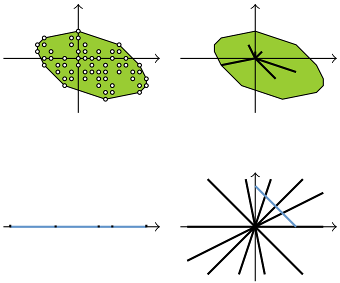

Thank you
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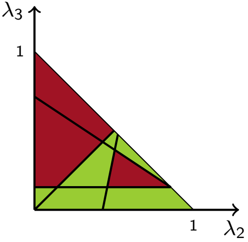

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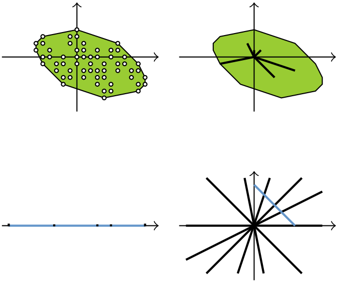

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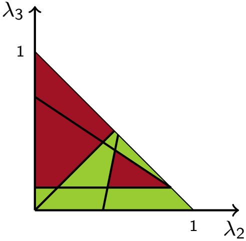
Questions?



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