

On Decision Uncertainty in Multiobjective Linear Programming

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Decision Uncertainty in Linear Objective Functions

A Multiobjective Robustness Gap for Decision Uncertainty in the Constraints



Decision Uncertainty: How the story started...

Example: Optimizing soil for potted plants in horticultural industries



Problem

What is a “good” mixing ratio for combining different kinds growing media for a plant?

Decision Variables mixing ratio per plant species

Objectives minimizing negative environmental impact, maximizing profit

- ▶ Putting a solution into practice causes problems:
 - ▶ technical limitations and workers will rather work fast than very exact
- ▶ realizations of decision variables are uncertain
- ▶ objective value of a realized decision variable is possibly not efficient

Decision Uncertainty in Multiobjective Optimization

Let

$$(P) \quad \begin{array}{ll} \min & C x \\ \text{s. t.} & A \cdot x \geq b \\ & x \in \mathbb{R}^N \end{array}$$

be a linear multiobjective optimization problem, where $C \in \mathbb{R}^{p \times n}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

To account for decision uncertainty in (P), we consider the corresponding uncertain problem:

$$\left\{ \begin{array}{ll} (P(z)) & \min \quad C(x+z) \\ & \text{s. t.} \quad A \cdot (x+z) \geq b \\ & \quad \quad \quad x \in \mathbb{R}^N \end{array} \right\}_{z \in Z} .$$

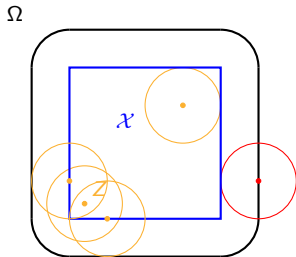


Decision Robust Feasibility

We consider

- ▶ a feasible set $\Omega \subset \mathbb{R}^n$
- ▶ a compact set $Z \subset \mathbb{R}^n$ with $0 \in Z$

- ▶ definition dates back to [Das97], [LP09].



Definition

A solution $x \in \Omega$ is called *decision robust feasible* with respect to Z if

$$x + z \in \Omega \quad \forall z \in Z.$$

We denote the set of *decision robust feasible solutions* by

$$\mathcal{X} := \bigcap_{z \in Z} \{x \in \mathbb{R}^n \mid x + z \in \Omega\} \subseteq \Omega.$$

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Decision Robust Efficiency

We consider

- ▶ a feasible set $\Omega \subset \mathbb{R}^n$
- ▶ a compact set $Z \subset \mathbb{R}^n$ with $0 \in Z$
- ▶ the decision robust feasible set $\mathcal{X} \subset \Omega$
- ▶ a continuous objective function $f: \Omega \rightarrow \mathbb{R}^p$
- ▶ the problem

$$(RC) \quad \min_{x \in \mathcal{X}} f_Z(x).$$

Definition

An element $x^* \in X$ is called *decision robust efficient* with respect to Z , if x^* is decision robust feasible and if there is no $x \in \mathcal{X} \setminus \{x^*\}$ with the property

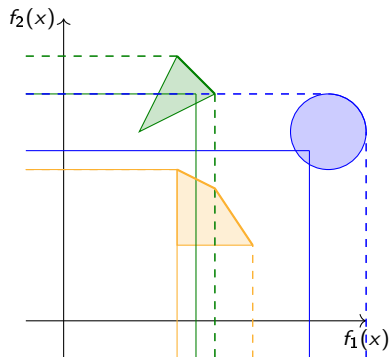
$$f_Z(x) \subseteq f_Z(x^*) - \mathbb{R}_+^p \setminus \{0\},$$

where $f_Z(x)$ is defined as the set $f_Z(x) = \{f(x+z): z \in Z\}$.



Decision Robustness in Multi-Objective Optimization

Bi-Objective



Decision Uncertainty in Linear Objective Functions

Theorem [EKS15]

Let $f: \Omega \rightarrow \mathbb{R}^p, x \mapsto Cx$ be linear. For each decision robust feasible solution $x \in \mathcal{X}$, the following two are equivalent:

$$\Leftrightarrow \begin{array}{ll} x \text{ is efficient for} & \text{(P}|\mathcal{X}\text{)} \quad \min_{x \in \mathcal{X}} Cx \\ x \text{ is decision robust efficient for} & \text{(RC)} \quad \min_{x \in \mathcal{X}} f_Z(x). \end{array}$$

- ▶ Also holds for closed convex pointed cones $K \subseteq \mathbb{R}^p$ instead of \mathbb{R}_+^p .

The Robust Counterpart for Decision Uncertainty in MOLP

For the uncertain problem

$$\left\{ \begin{array}{ll} \text{(P(z))} & \min \quad C(x+z) \\ & \text{s. t.} \quad A \cdot (x+z) \geq b \\ & \quad \quad \quad x \in \mathbb{R}^n \end{array} \right\}_{z \in Z} .$$

We hence obtain the *Robust Counterpart*

$$\text{(RC)} \quad \min \quad Cx \\ \text{s. t.} \quad A(x+z) \geq b \quad \forall z \in Z \\ \quad \quad \quad x \in \mathbb{R}^n$$

We rearrange the side conditions

$$\Leftrightarrow \begin{array}{ll} A(x+z) \geq b & \forall z \in Z \\ Ax \geq b - Az & \forall z \in Z \end{array}$$

Define a different uncertainty set

$$\mathcal{U} := \{\xi = b - Az \mid z \in Z\} \subseteq \mathbb{R}^m$$

We finally obtain the *Robust Counterpart* as

$$\text{(RC)} \quad \min \quad Cx \\ \text{s. t.} \quad Ax \geq \xi \quad \forall \xi \in \mathcal{U} \\ \quad \quad \quad x \in \mathbb{R}^n$$

The Robust Counterpart for Decision Uncertainty in MOLP

$$\begin{aligned} \text{(RC)} \quad & \min && Cx \\ & \text{s. t.} && Ax \geq \xi \quad \forall \xi \in \mathcal{U} \\ & && x \in \mathbb{R}^n \end{aligned}$$

- ▶ In addition to assuming Z compact, we also require Z to be a convex polyhedral set.
- ▶ Hence, the uncertainty set $\mathcal{U} := \{\xi = b - Az \mid z \in Z\} \subseteq \mathbb{R}^m$ is a compact convex polyhedral set as well.
- ▶ Then, there exist $D \in \mathbb{R}^{s \times m}$ and $d \in \mathbb{R}^s$ such that

$$\mathcal{U} = \{\xi \in \mathbb{R}^m \mid D\xi \leq d\}.$$

- ▶ In the following, we study MOLP with deterministic objective functions and polyhedral uncertainty in the right hand side of the constraints.
- ▶ The side conditions of (RC) fit into the robust optimization framework of [BTGN09].

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The Robustness Gap

- ▶ Decision makers often want to know how much they “sacrifice” for gaining a robust efficient solution.
- ▶ How to measure the different performance of an efficient and a robust efficient solution?

In single-objective optimization, we refer to the well-known concept of [BTGN09].

Definition

Let $g: \Omega \rightarrow \mathbb{R}$, let an uncertain linear optimization problem (P^{s-o}) and its robust counterpart be given as

$$(P^{s-o}) \quad \{\min\{g(x) \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}} \quad \text{and} \quad (RC^{s-o}) \quad \min\{g(x) \mid Ax \geq \xi, \forall \xi \in \mathcal{U}\}.$$

The *single-objective robustness gap* is defined as

$$RGSO := \min_{x: Ax \geq \xi \forall \xi \in \mathcal{U}} g(x) - \sup_{\xi \in \mathcal{U}} \min_{x: Ax \geq \xi} g(x).$$

Example: Robustness Gap

Let

$$\begin{aligned} (P^{s-o}) \quad & \min && \frac{1}{\sqrt{5}}(x_1 + 2x_2) \\ & \text{s. t.} && x_1 \geq \xi_1 \\ & && x_2 \geq \xi_2 \quad \forall \xi \in \mathcal{U} \\ & && x_1 + x_2 \geq \xi_3 \end{aligned}$$

where $\mathcal{U} = \left\{ \begin{pmatrix} 10 \\ 0 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 15 \end{pmatrix} \right\}$.

The corresponding robust counterpart is

$$\begin{aligned} (RC^{s-o}) \quad & \min && \frac{1}{\sqrt{5}}(x_1 + 2x_2) \\ & \text{s. t.} && x_1 \geq 10 \\ & && x_2 \geq 10 \\ & && x_1 + x_2 \geq 15 \end{aligned}$$

The robustness gap is

$$RGSO = \min_{x: Ax \geq \xi \forall \xi \in \mathcal{U}} g(x) - \sup_{\xi \in \mathcal{U}} \min_{x: Ax \geq \xi} g(x) = \frac{30}{\sqrt{5}} - \frac{25}{\sqrt{5}} = \sqrt{5}.$$



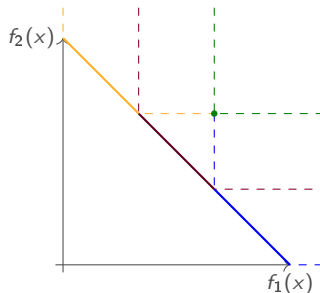
Example: Robustness Gap

Consider a multiobjective problem with uncertain right hand-sides:

$$\begin{aligned} (P) \quad & \min && \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ & \text{s. t.} && x_1 \geq \xi_1 \\ & && x_2 \geq \xi_2 \\ & && x_1 + x_2 \geq \xi_3 \end{aligned} \quad \forall \xi \in \mathcal{U}$$

$$\text{where } \mathcal{U} = \left\{ \begin{pmatrix} 10 \\ 0 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 15 \end{pmatrix} \right\}.$$

How can we define a robustness gap in multiple dimensions?



Defining a Robustness Gap for MOLP with Decision Uncertainty

Definition

Let $\{(P(\xi)) \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. We denote the set of decision robust efficient solutions to (RC) by \mathcal{Y}_P^{RC} and for each $\xi \in \mathcal{U}$ we denote the set of efficient solutions to $P(\xi)$ by $\mathcal{Y}_P(\xi)$. We define the *Robustness Gap* for $\{(P(\xi))\}_{\xi \in \mathcal{U}}$ as

$$\vartheta := \min_{\xi \in \mathcal{U}} \left(\min_{y \in \mathcal{Y}_P(\xi)} \min_{z \in \mathcal{Y}_P^{RC}} \|y - z\|_2 \right).$$

- ▶ Computing the robustness gap ϑ requires optimizing over the Pareto-sets of the problems $(P(\xi))$.
- ▶ We want to approximate the robustness gap ϑ .
- ▶ lower and upper bounds on ϑ ?
- ▶ Obviously, $\vartheta \geq 0$

A lower bound on the robustness gap \mathcal{V}

- ▶ We make use of the single-objective definition of the robustness gap by [BTGN09].
- ▶ By weighted-sum scalarization, we transform our multiobjective problem $\{(P(\xi))\}$ into a single-objective problem.

Definition

Let $\{(P(\xi)) \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. For each $\xi \in \mathcal{U}$ and for each $\lambda \in \mathbb{R}_+^p$, we define

$$\Delta(\xi, \lambda) := \min_{z \in \mathcal{Y}_P^{RC}} \lambda^T z - \min_{y \in \mathcal{Y}_P(\xi)} \lambda^T y.$$

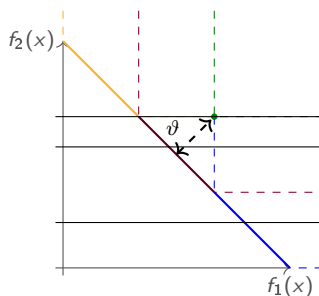
Furthermore, the *lower Robustness Gap* is defined as

$$\Delta := \min_{\xi \in \mathcal{U}} \min_{\substack{\lambda \in \mathbb{R}_+^p, \\ \|\lambda\|=1}} \Delta(\xi, \lambda).$$

A lower bound on the robustness gap ϑ

In our example, we have

$$\Delta = 0 < \sqrt{2} \cdot 5 = \vartheta.$$



Theorem

Let $\{(P(\xi)) \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. The lower robustness gap Δ is indeed a lower bound for the robustness gap ϑ , i.e.,

$$\begin{aligned} \Delta &= \min_{\xi \in \mathcal{U}} \min_{\substack{\lambda \in \mathbb{R}_+^p, \\ \|\lambda\|=1}} \left(\min_{z \in \mathcal{Y}_P^{RC}} \lambda^T z - \min_{y \in \mathcal{Y}_P(\xi)} \lambda^T y \right) \\ &\leq \vartheta = \min_{\xi \in \mathcal{U}} \left(\min_{y \in \mathcal{Y}_P(\xi)} \min_{z \in \mathcal{Y}_P^{RC}} \|y - z\|_2 \right). \end{aligned}$$

The Relationship between the RG ϑ and its lower bound Δ

- In which cases is lower bound Δ a tight approximation of the robustness gap ϑ ?

Let $\{(P(\xi)) \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty.

Theorem

Let there exist $\bar{z} \in \mathcal{Y}_P^{RC}$, $\bar{\xi} \in \mathcal{U}$ and $\bar{y} \in \mathcal{Y}_P(\bar{\xi})$ that satisfy

$$\vartheta = \|\bar{z} - \bar{y}\|_2$$

such that there exists a $p - 1$ -dimensional face \mathcal{F} of the polyhedral set $\mathcal{Y}(\xi)$ with $\bar{y} \in \text{relInt}(\mathcal{F})$.

Then Δ a tight approximation of the robustness gap ϑ , i.e.,

$$\Delta = \vartheta.$$

The Relationship between the RG ϑ and its lower bound Δ

Let $\{(P(\xi)) \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty.

Theorem

Let there exist $\bar{z} \in \mathcal{Y}_P^{RC}$, $\bar{\xi} \in \mathcal{U}$ and $\bar{y} \in \mathcal{Y}_P(\bar{\xi})$ that satisfy

$$\vartheta = \|\bar{z} - \bar{y}\|_2$$

such that there exists a $p - 1$ -dimensional face \mathcal{F} of the polyhedral set \mathcal{Y}^{RC} with $\bar{z} \in \text{relInt}(\mathcal{F})$.

Then Δ a tight approximation of the robustness gap ϑ , i.e.,

$$\Delta = \vartheta.$$

- ▶ Furthermore, $\Delta = \vartheta$ whenever the robustness gap ϑ is 0.
- ▶ How to determine the lower bound Δ ?

Determining the lower bound Δ by Quadratic Programming

Using the assumption $\mathcal{U} = \{\xi \in \mathbb{R}^m \mid D\xi \leq d\}$, we obtain by the dual formulations of (P(ξ)) and (RC):

$$\begin{aligned} \Delta = \quad & \min && \bar{b}^T u - \xi^T v \\ & \text{s. t.} && v^T A = \lambda^T C \\ & && D\xi \leq d \\ & && u^T A = \lambda^T C \\ & && Ax \geq \bar{b} \\ & && \lambda^T Cx = \bar{b}^T u \\ & && \|\lambda\|_2 = 1 \\ & && v, u, \lambda \geq 0 \end{aligned}$$

Define

$$\bar{b} = \left\{ \max_{\xi \in \mathcal{U}} \xi_i \right\}_{i=1}^m \in \mathbb{R}^m.$$

Thank you for your attention!



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