Primal and Dual Methods for Optimisation over the Non-dominated Set of a Multi-objective Programme and Computing the Nadir Point

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Outline

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Multi-objective Optimisation Problems

A multi-objective optimisation problem is formulated as

$$\min \{f(x) : x \in \mathcal{X}\}, \tag{MOP}$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{X}$ is a feasible set. we assume that $\mathcal{X}$ is a nonempty and compact set.

$$\min \{Cx : x \in \mathcal{X}\} \tag{MOLP}$$

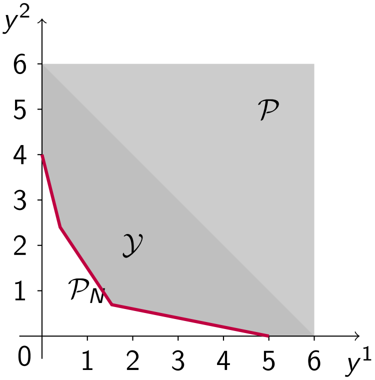
- C is a $p \times n$ matrix, whose rows are the coefficients of objective functions $c^k x$, $k = 1, \ldots, p$.
- $\mathcal{X} := \{x \in \mathbb{R}^n : Ax \geqq b\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- $\mathcal{Y} := \{y \in \mathbb{R}^p : y = Cx, x \in \mathcal{X}\}$ is the feasible set in objective space $\mathbb{R}^p$.
- $\mathcal{P} := \mathcal{Y} + \mathbb{R}^p_{\geqq}$ is the extended feasible set in objective space $\mathbb{R}^p$.

Example

$$\min \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 1 & 5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geqq \begin{pmatrix} 4 \\ 6 \\ 5 \\ -6 \end{pmatrix}$$

$$x_1, x_2 \geqq 0.$$



Optimisation over the Non-dominated Set

(ON)

$$\max \{M(y) : y \in \mathcal{P}_N\}, \qquad \text{(ON)}$$

where $\mathcal{P}_N$ is the non-dominated set of a multi-objective optimisation problem.

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(P)

The simplest form of (ON) is

$$\max \{\mu^T y : y \in \mathcal{P}_N\}. \qquad \text{(P)}$$

Motivation

1. The burden of generating the entire set of non-dominated points may be saved.
2. Decision makers may be overwhelmed by the immensity of the whole non-dominated set and may not be able to choose a preferred solution from it.

Challenges

1. The feasible set of (P), $\mathcal{P}_N$, is a nonconvex set even in the case of (MOLP).
2. The feasible set of (P) problems, $\mathcal{P}_N$, can not be expressed in the format as a system of inequalities.
3. Mathematically, problem (P) is a difficult global optimisation problem, the number of local optima can be very large.

Outline

Decision Space Based Algorithms

Adjacent vertex search algorithms Philip (1972), Ecker and Song (1994), Fülöp (1994), Bolintineanu (1993)

Nonadjacent vertex search algorithm Benson (1992)

Face search algorithm Sayin (2000)

Lagrangian relaxation method White (1996), Dauer and Fosnaugh (1995) and An et al. (1996)

Duality method Tuyen and Muu (2001)

Branch and bound method Yamada et al. (2000) Yamada et al. (2001)

Bisection method Thai Quynh and Hoang Quang (2000)

Global and Local search Le Thi et al. (2002)

Objective Space Based Algorithms

Bi-objective branch and bound algorithm Fülöp and Muu (2000)

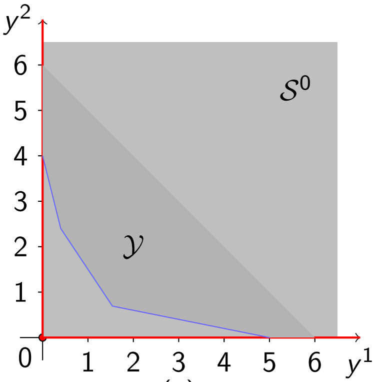
Conical branch and bound algorithm Thoai (2000)

Benson's branch and bound algorithm Benson (2011)

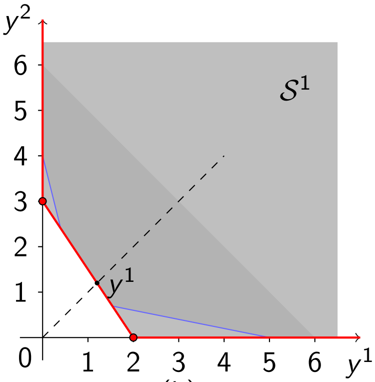
Outline

Revised version of Benson's Algorithm to Solve (MOLP) in primal objective space (Löhne et al. (2014))

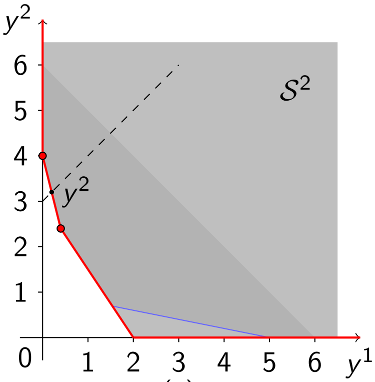
1: Compute the optimal value y_i^I of $(P_1(e^i))$, for $i = 1, \dots, p$.
$Y_N := \emptyset$, for $i = 1, \dots, p$.
2: Set $\mathcal{S}^0 := \{y^I\} + \mathbb{R}^p_{\geqq}$; $V_{\mathcal{S}^0} := \{y^I\}$ and $k := 1$.
3: **while** $V_{\mathcal{S}^{k-1}} \not\subset \mathcal{P}$ **do**
4: Choose a vertex s^k of $\mathcal{S}^{k-1}$ such that $s^k \notin \mathcal{P}$.
5: Compute an optimal solution (x^k, z^k) to $P_2(s^k)$ and its dual variable (u^k, λ^k).
6: Set
$\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{y \in \mathbb{R}^p : \varphi(y, (\lambda_1^k, \dots, \lambda_{p-1}^k, b^T u^k)) \geqq 0\}$;
Update $V_{\mathcal{S}^{k-1}}$; $Y_N := Y_N \cup Cx^k$.
7: Set $k := k + 1$
8: **end while**



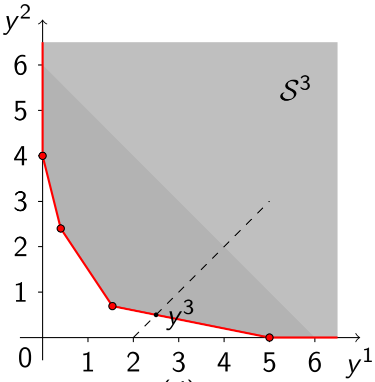
(a)



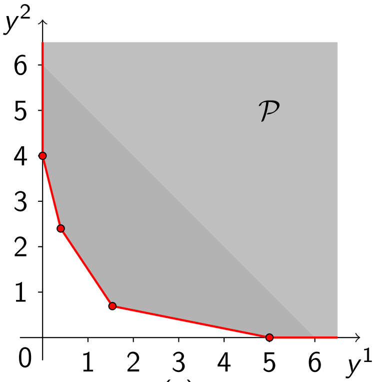
(b)



(c)



(d)

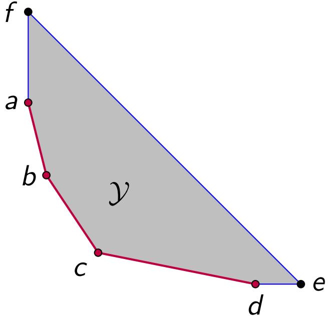


(e)

Properties of (P)

Theorem

The optimal solution of problem (P) is obtained at a vertex of $\mathcal{P}_N$, i.e., $y^* \in V_{\mathcal{P}_N}$.

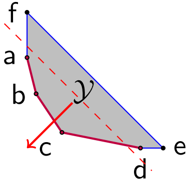


Properties of (P)

Theorem

(P) has the same optimal solution as (RP), if and only if $\mu \in \mathbb{R}^p_{\leqq}$.

$$\max \{\mu^T y : y \in \mathcal{P}\} \ (RP)$$



Properties of (P)

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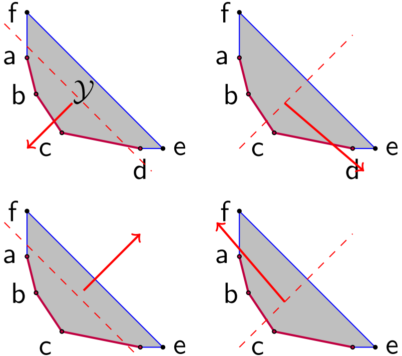
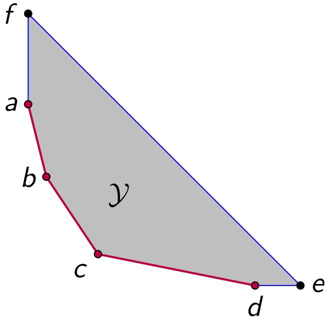


Properties of (P)

Theorem

For all $\mu \in \mathbb{R}^p \setminus \mathbb{R}^p_{\leqq}$, an optimal solution y^* of (P) is attained at an incomplete non-dominated vertex, i.e., there is $y^* \in V^{ic}_{\mathcal{Y}} \cap V_{\mathcal{P}}$.



Primal Method

Initialization

1: *Initialization ($k = 0$).*
2: Compute the optimal value y_i^I of $(P_1(e^i))$, for $i = 1, \ldots, p$.
3: Set $\mathcal{S}^0 := \{y^I\} + \mathbb{R}^p_{\geqq}$, $k := 1$ and $V_{\mathcal{S}^0} := \{y^I\}$.
4: Threshold := False.
5: *Iteration steps ($k \geqq 1$).*

Primal Method

Iterations

while $V_{\mathcal{S}^{k-1}} \not\subset \mathcal{P}$ **do**
  $s^k :\in \text{argmax}\{\mu^T y : y \in V_{\mathcal{S}^{k-1}}\}$,
  if $s^k \in \mathcal{P}$ and Threshold $=$ False **then**
    $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{y \in \mathbb{R}^p : \mu^T y \geqq \mu^T s^k\}$. Update $V_{\mathcal{S}^{k-1}}$.
    Threshold := True.
  else
    Compute an optimal solution (x^k, z^k) to $P_2(s^k)$ and its dual optimal variable values (u^k, λ^k).
    Set
    $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{y \in \mathbb{R}^p : \varphi(y, (\lambda_1^k, \ldots, \lambda_{p-1}^k, b^T u^k)) \geqq 0\}$.
    Update $V_{\mathcal{S}^{k-1}}$.
  end if
  Set $k := k + 1$.
end while

Example

$$\max: \; y_1 + y_2 \text{ s.t. } \; y \in \mathcal{P}_N,$$

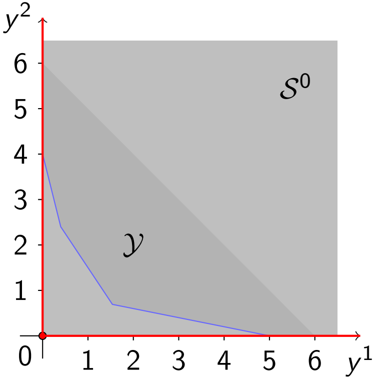
where $\mathcal{P}_N$ is the extended non-dominated set of the following problem

$$\min \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

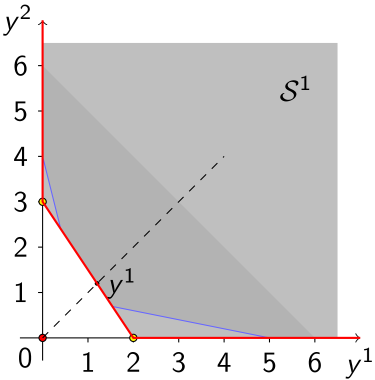
$$\text{s.t. } \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 1 & 5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geqq \begin{pmatrix} 4 \\ 6 \\ 5 \\ -6 \end{pmatrix}$$

$$x_1, x_2 \geqq 0.$$

Example



Example



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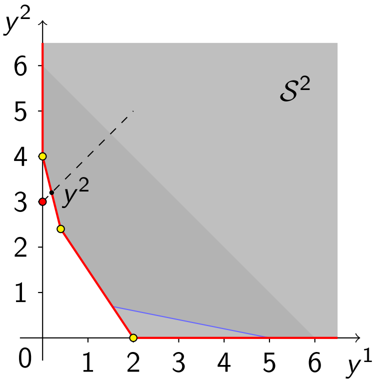

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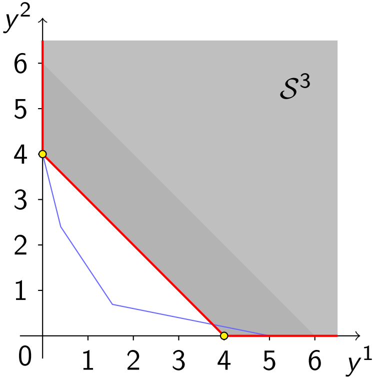

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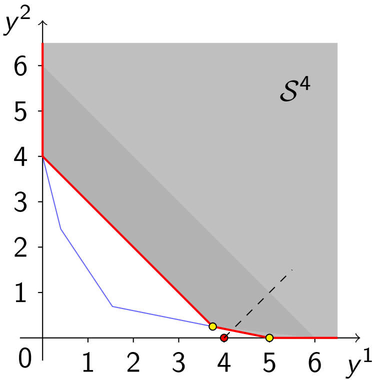


Outline

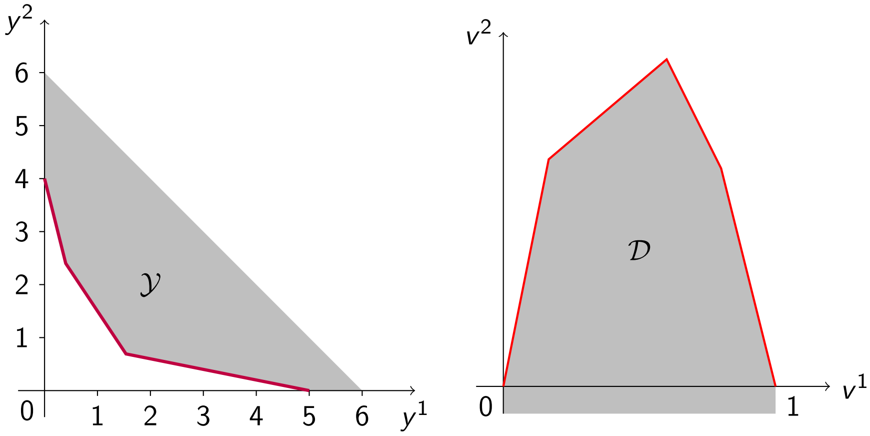
Geometric dual of (MOLP)

The dual of (MOLP) is

$$\max_{\mathcal{K}} \{(\lambda_1, ..., \lambda_{p-1}, b^T u)^T : (u, \lambda) \geqq 0, A^T u = C^T \lambda, e^T \lambda = 1\}, \tag{DMOLP}$$

where $(u, \lambda) \in \mathbb{R}^m \times \mathbb{R}^p$.
$\mathcal{K} := \{v \in \mathbb{R}^p : v_1 = v_2 = \cdots = v_{p-1} = 0, v_p \geqq 0\}$ is the ordering cone in the dual objective space, and maximisation is with respect to the order defined by $\mathcal{K}$.

Example



Geometric Duality (Heyde and Löhne (2008))

$$\lambda(v) := \left(v_1, \ldots, v_{p-1}, 1 - \sum_{i=1}^{p-1} v_i\right)^T$$

$$\lambda^*(y) := (y_1 - y_p, \ldots, y_{p-1} - y_p, -1)^T .$$

Using the coupling function φ, let's define the following two set-valued maps

$$H : \mathbb{R}^p \rightrightarrows \mathbb{R}^p, H(v) := \{y \in \mathbb{R}^p : \lambda(v)^T y = v_p\}.$$

$$H^* : \mathbb{R}^p \rightrightarrows \mathbb{R}^p, H^*(y) := \{v \in \mathbb{R}^p : \lambda^*(y)^T v = -y_p\}.$$

Geometric Duality (Heyde and Löhne (2008))

The following theorems are essential to the primal and dual methods in the subsequent sections.

Theorem

Heyde and Löhne (2008) The following statements are equivalent
(i) v is a $\mathcal{K}$-maximal vertex of $\mathcal{V}$.
(ii) $H(v) \cap \mathcal{P}$ is a weakly non-dominated facet of $\mathcal{P}$.

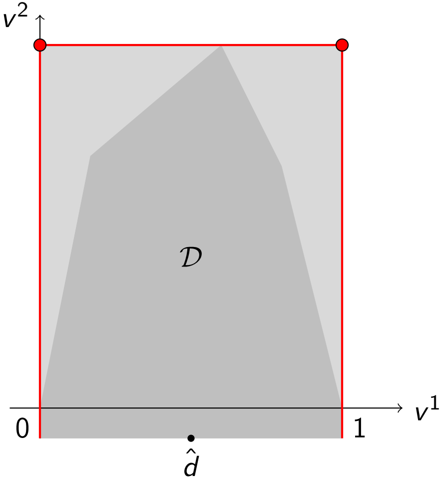
Theorem

(Heyde and Löhne (2008)) The following statements are equivalent
(i) y is a is a weakly non-dominated vertex of $\mathcal{P}$.
(ii) $H^*(y) \cap \mathcal{V}$ is a $\mathcal{K}$-maximal facet of $\mathcal{V}$.

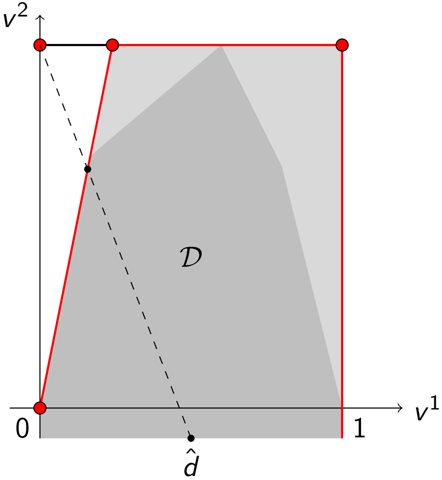
Dual Variant of Benson's Algorithm (Löhne et al. (2014))

Choose some $\hat{d} \in int\mathcal{D}$.
Compute an optional solution x^0 of $P_1(\hat{d})$.
Set $S^0 := \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(Cx^0, v) \geqq 0\}$ *and* $k = 1$.
Iteration steps $(k \geqq 1)$
while vert$\mathcal{S}^{k-1} \subsetneq \mathcal{V}$ **do**
    choose a vertex s^k of $\mathcal{S}^{k-1}$ such that $s^k \notin \mathcal{V}$.
    Compute $\alpha^k \in (0, 1)$ such that
    $v^k := \alpha^k s^k + (1 - \alpha^k)\hat{d} \in \mathcal{V}_{\mathcal{K}}$.
    Compute an optimal solution x^k of $(P_1(v^k))$.
    Set $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{v \in \mathbb{R} : \varphi(Cx^k, v) \geqq 0\}$.
    Set $k := k + 1$.
end while

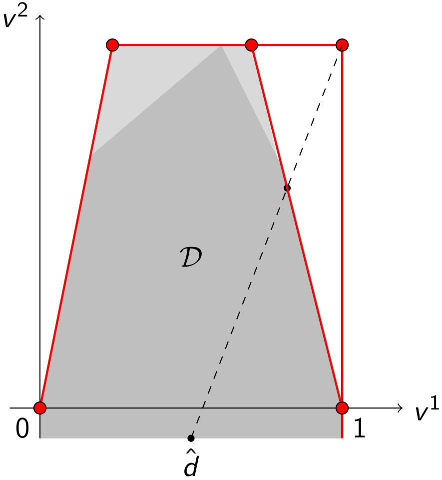
Example



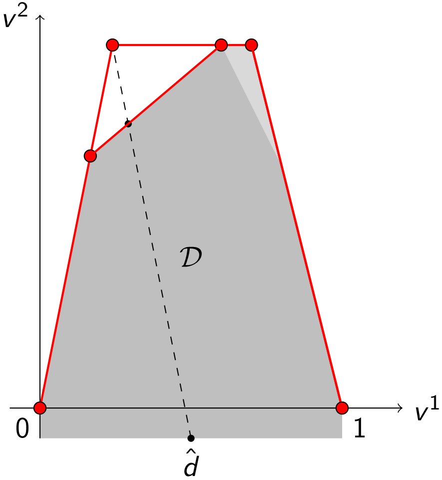
Example



Example



Example



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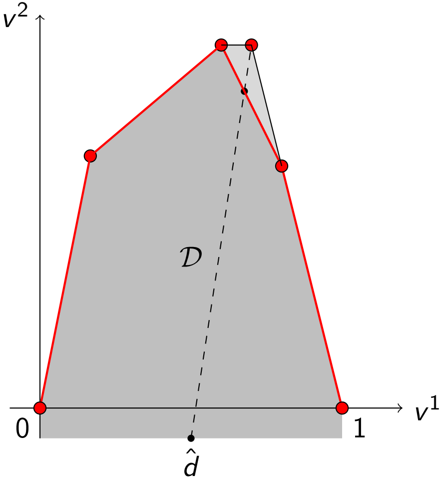
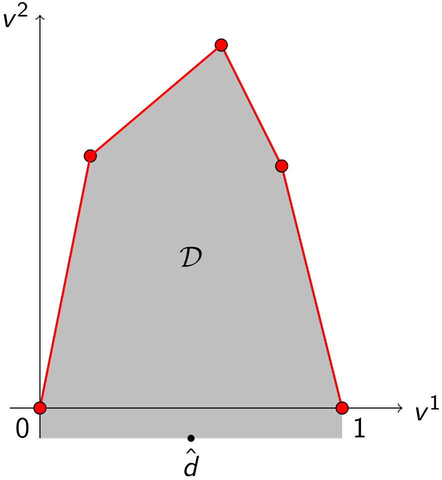


Example



Properties of (P) in dual objective space

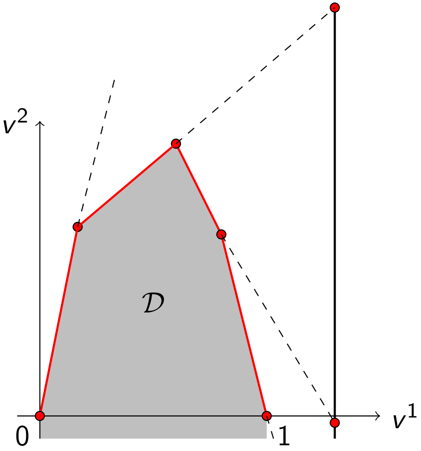
$$L^{\mu} := \{v \in \mathbb{R}^p : v_1 = v_1^{\mu}, ..., v_{p-1} = v_{p-1}^{\mu}\}.$$

Theorem

v^{μ} lies on $H^*(y^{ex})$

Therefore it is unnecessary to obtain the complete $\mathcal{K}$-maximal hyperplane set. Because we are just interested in finding the "highest" intersection point, i.e., the point with the largest last element value, between the $\mathcal{K}$-maximal hyperplanes and L^{μ}.

Example



Neighbouring relation

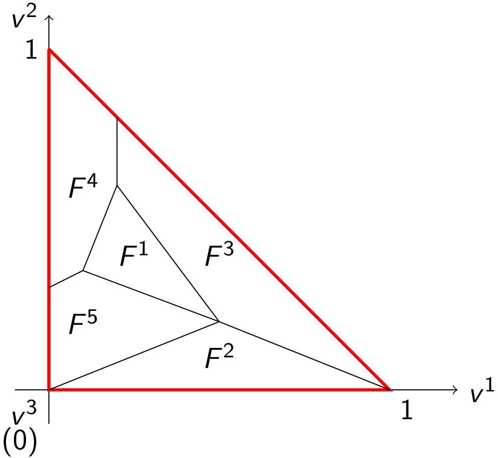
Definition

Two facets $\mathcal{K}$-maximal F^i and F^j of polyhedron $\mathcal{D}$ are called neighbouring facets if $dim(F^i \cap F^j) = p - 2$.

Proposition

If y^i, $y^j \in V_{\mathcal{P}}$, *and* y^i *and* y^j *are neighbours in primal space.*
facets $F^i = H^*(y^i) \cap \mathcal{D}$ *and* $F^j = H^*(y^i) \cap \mathcal{D}$ *are neighbours.*

Projection of a three dimensional dual polyhedron onto the v^1-v^2 coordinate plane



Properties of (P) in dual objective space

Definition

If all the neighbours of a facet are $\mathcal{K}$-maximal facets, then this facet is called a complete facet denoted by F^c. Otherwise, it is called an incomplete facet denoted by F^{ic}.

Theorem

A complete facet of $\mathcal{D}$ corresponds to a complete vertex of $\mathcal{P}$ and vice versa. An incomplete facet of $\mathcal{D}$ corresponds to an incomplete vertex of $\mathcal{P}$ and vice versa.

Dual method to solve (P)

Choose some $\hat{d} \in \mathrm{int}\mathcal{V}$.
Compute an optimal solution x^0 of $P_1(\hat{d})$, $M^0 = \mu^T C x^0$.
Set $S^0 := \{v \in \mathbb{R}^p : \lambda(v) \geqq 0, \varphi(Cx^0, v) \geqq 0\}$ *and* $k = 1$.
Iteration steps $(k \geqq 1)$
while $W_D \cap V_{\mathcal{S}^{k-1}} \neq \emptyset$ **do**
  Choose $v \in W \cap V_{\mathcal{S}^{k-1}}$ such that $v \notin \mathbb{V}$.
  Compute $\alpha^k \in (0,1)$ such that $v^k := \alpha^k s^k + (1-\alpha^k)\hat{d} \in \mathbb{V}_{\mathcal{K}}$.
  Compute an optimal solution x^k of $(P_1(v^k))$, $M^k = \mu^T C x^k$.
  if $M^k < M^{k-1}$ **then**
    $M^* = M^k$
  end if
  Set $\mathcal{S}^k := \mathcal{S}^{k-1} \cap \{v \in \mathbb{R} : \varphi(Cx^k, v) \geqq 0\}$.
  Set $k := k + 1$.
end while

Example

$$\text{max: } y_1 + y_2 \text{ s.t. } y \in \mathcal{P}_N,$$

where $\mathcal{P}_N$ is the non-dominated set of the following problem

$$\min \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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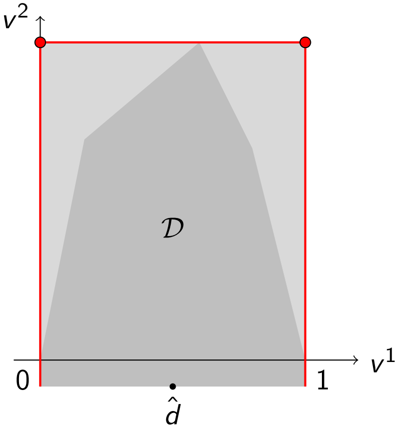
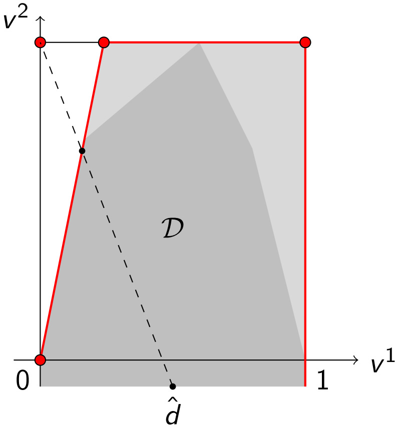
$$x_1, x_2 \geqq 0.$$

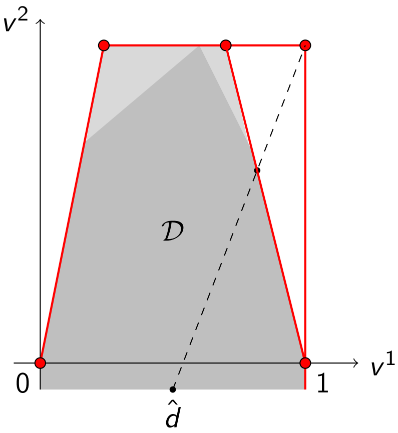
Example



Example



Example



Outline

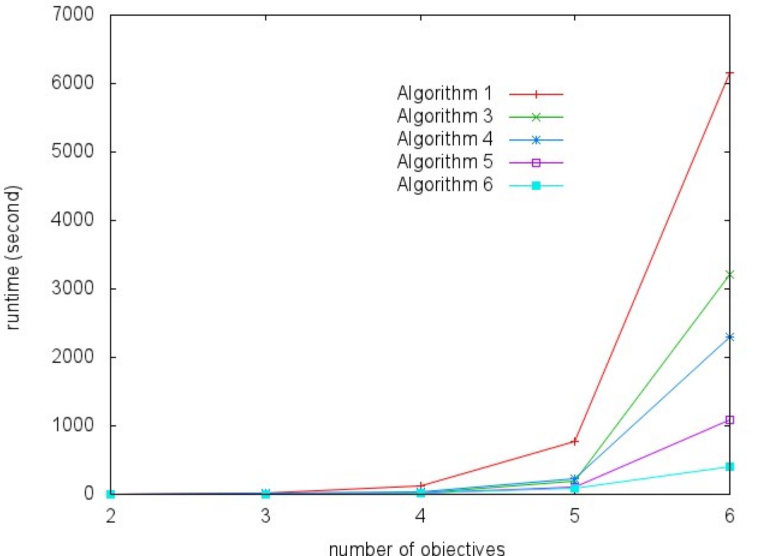
Computational Experiments

1. Brute force algorithm
2. Bi-objective branch and bound
3. Conical branch and bound
4. Benson's branch and bound
5. Primal method
6. Dual method

Computational Experiments

p	m, n	1	2	3	4	5	6
	5	0.2944	0.0064	0.0204	0.0973	0.0098	0.0269
	10	0.2271	0.0083	0.0186	0.1067	0.0097	0.0253
2	50	0.2630	0.0137	0.0746	0.6703	0.0386	0.0484
	100	0.3538	0.0263	0.1096	3.6720	0.0661	0.1007
	500	4.2958	0.5186	2.9720	5.2351	2.9951	1.6861
	5	0.2257	-	0.0014	0.1657	0.0017	0.0417
	10	0.3207	-	0.0502	0.2557	0.0395	0.0483
3	50	0.5055	-	0.1327	1.5828	0.0936	0.0868
	100	0.8833	-	0.9113	3.4237	0.7834	0.2022
	500	11.0789	-	6.2259	9.3610	6.2472	2.9464
	5	0.6030	-	0.1265	0.8088	0.1120	0.0743
	10	0.5747	-	0.0014	0.9342	0.0015	0.1408
4	50	0.8470	-	0.4054	5.8732	0.3865	0.1948
	100	3.1461	-	2.7900	10.4872	2.3220	0.4795
	500	114.8987	-	21.7083	39.1832	20.8440	16.2388
	5	1.0357	-	1.0292	4.9149	0.9284	0.1124
	10	1.8133	-	1.0383	7.3492	0.7925	0.2116
5	50	7.2347	-	3.1660	13.8473	2.3183	0.3775
	100	25.9615	-	10.6977	74.4837	7.5761	2.4312
	500	764.4023	-	186.3596	230.4837	107.9541	80.7296
	5	17.7494	-	0.6792	0.4837	0.6356	0.4111
	10	33.7749	-	42.2668	50.3844	31.3130	0.4561
6	50	307.6071	-	44.1350	70.3344	34.5798	2.0074
	100	2802.8769	-	123.9998	133.8543	33.2809	8.4988
	500	6165.0527	-	3209.3518	2302.4857	1081.2941	397.7225

Results of instances with m and n equal 500

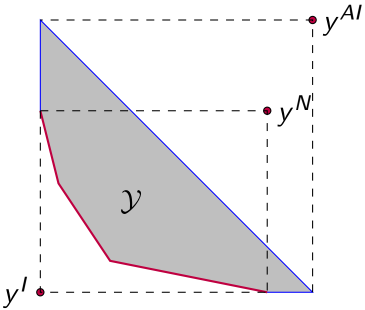


Outline

y^I, y^N and y^{AI}

The nadir point $y^N \in \mathbb{R}^p$ is characterized by the componentwise maximal values of non-dominated points for (MOP), i.e.,

$$y_k^N := \max \{f^k(x) : x \in \mathcal{X}_E\} \quad k = 1, ..., p.$$



Using y^N as a reference point

Compromise programming

$$\max \{||f(x) - y^N|| : x \in \mathcal{X}\}.$$

Primal method for nadir point

1: $N := \emptyset$.
2: Compute $\hat{x}^k \in \text{argmin}\{c^k x : x \in \mathcal{X}\}$.
3: Compute $x^k \in \text{argmin}\{\sum_{j \neq k} c^j x : x \in \mathcal{X}, c^k x = c^k \hat{x}\}$.
$N = N \cup \{Cx^k\}$.
4: Set $\mathcal{S}^0 := \{y^I\} + \mathbb{R}^p_{\geqq}$.
5: $y_k := \max\{y_k : y \in N\}$.
6: $\mathcal{S}^1 := \mathcal{S}^0 \cap \{y \in \mathbb{R}^p : e^{kT} y \geqq y_k\}$. Threshold = True. $i = 2$.

Primal method for nadir point

while $V_{\mathcal{S}^{k-1}} \not\subset \mathcal{P}$ **do**

  $s^i := \max\{y_k : y \in N \cup V_{\mathcal{S}^{i-1}}\}$.

  if $s^i \in \mathcal{P}$ and if Threshold $=$ False **then**

    $\mathcal{S}^i := \mathcal{S}^{i-1} \cap \{y \in \mathbb{R}^p : e^{kT} y \geqq s^i_k\}$. Threshold $=$ True.

  else

    Compute an optimal solution (x^i, z^i) to $P_2(s^i)$ and its dual variable values (u^i, λ^i). $N = N \cup \{Cx^i\}$.

    Set $\mathcal{S}^i := \mathcal{S}^{i-1} \cap \{y \in \mathbb{R}^p : \varphi(y, (\lambda^i_1, \ldots, \lambda^i_{p-1}, b^T u^i)) \geqq 0\}$.

  end if

  $N = N \cup (V_{\mathcal{S}^i} \cap \mathcal{P})$.

  Set $i := i + 1$.

end while

$y^N_k = s^i_k$.

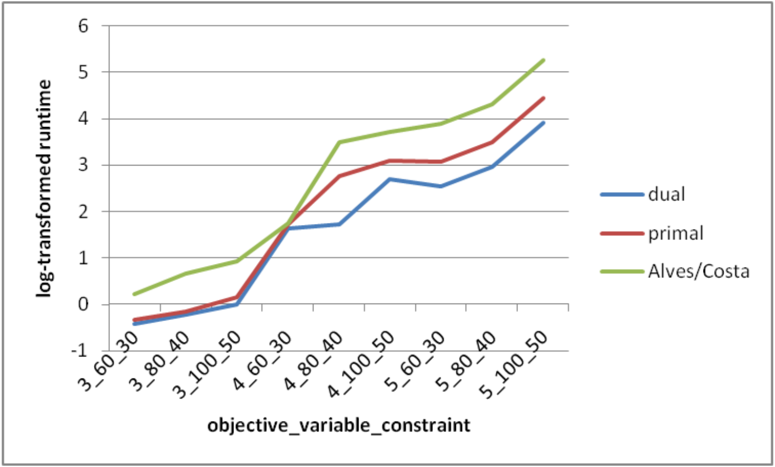
Dual method for nadir point

1: Perform the dual method to obtain the incomplete vertices.
2: **for** $k = 1, \ldots, p$ **do**
3: $y_k^N = \max \{e^{kT} y : y \in Y\}$.
4: **end for**

Computational Experiments

- Primal algorithm
- Dual algorithm
- LP-based exact algortihm by Alves and Costa (2009)
- 3 – 5 objectives, 60-100 variables, 30 – 50 constraints

Computational Experiments



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