

Multi-objective decision uncertainty and set optimization with the set approach

Gabriele Eichfelder

joint work with Corinna Krüger, Anita Schöbel, and Maria Pilecka

Technische Universität Ilmenau

Lancaster

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Content

1. Decision Uncertainty
2. Multi-Objective Decision Robust Efficiency
3. Relation to Set Optimization
4. (Numerical) Approaches to Set Optimization



Section 1

Decision Uncertainty

Basic idea

Given: function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, a feasible set $\Omega \subset \mathbb{R}^n$,
and the optimization problem

$$\bar{x} \in \operatorname{argmin}_{x \in \Omega} f(x)$$

but ...

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... \bar{x} cannot be put into practice exactly
(e.g. the optimal temperature/height/volume)
 \Rightarrow Uncertainty in the realization

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⇒ Uncertainty in the realization

... . . . we choose a robust approach (in the sense of Ben-Tal and Nemirovski): "min-max"

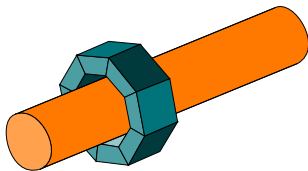
Example: Design of a magnetic system

The Lorentz force velocimetry (LFV) is an electromagnetic non-contact flow measurement technique for electrically conducting fluids.

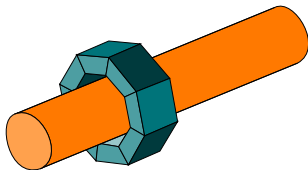
The task is to find an optimal magnet design which

- ▶ results in a strong Lorentz force (f_1)
- ▶ and has a minimal weight (f_2)

i.e. we have **two** objective functions.



Example: Design of a magnetic system



- ▶ variables: direction of the magnetization (angles Φ_i and Θ_i) and the magnetization M_i of each magnet i
- ▶ in practice an (optimally) chosen magnetic direction $\bar{\Phi}_i, \bar{\Theta}_i$ cannot be realized in the desired accuracy
- ▶ only magnets can be ordered with a guarantee of $\Phi_i \in [\bar{\Phi}_i - \varepsilon, \bar{\Phi}_i + \varepsilon]$ and $\Theta_i \in [\bar{\Theta}_i - \varepsilon, \bar{\Theta}_i + \varepsilon]$ for some $\varepsilon > 0$.

Notation and Assumptions

Given: compact set Z of all possible deviations (with $0 \in Z$)
instead of x we have to consider $\{x + z \mid z \in Z\}$

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instead of x we have to consider $\{x + z \mid z \in Z\}$

Given: $f: \mathbb{R}^n \rightarrow \mathbb{R}$ continuous,
and a feasible set $\Omega \subseteq \mathbb{R}^n$

Consider for each $x \in \Omega$ the set

$$f_Z(x) := \{f(x + z) \in \mathbb{R} \mid z \in Z\}$$

instead of $f(x)$.

Feasibility

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and feasible set $\Omega \subseteq \mathbb{R}^n$

Definition (Feasibility, Lewis '02)

A point $x \in \Omega$ is called *decision robust feasible* with respect to Z if

$$\{x + z \mid z \in Z\} \subseteq \Omega,$$

i.e., if all realizations of x are feasible.

Feasible set of the robust problem:

$$X := \{x \in \Omega \mid x + z \in \Omega \text{ for all } z \in Z\}$$

(Scalar-valued) Decision Uncertainty

Under the name robust regularization Lewis suggested for a scalar-valued objective function to solve

$$\min_{x \in X} \sup_{z \in Z} f(x + z)$$

i.e. for the map

$$x \mapsto \bar{f}(x) := \sup_{z \in Z} f(x + z)$$

to solve the problem

$$\min_{x \in X} \bar{f}(x).$$

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But what to do in case of $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m > 1$?



Section 2

Multi-Objective Decision Robust Efficiency

Efficient solutions in multi-objective optimization

Consider

$$\min_{x \in \Omega} f(x) \quad \text{with} \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

and a pointed convex cone $K \subset \mathbb{R}^m$ (with $\text{int}(K) \neq \emptyset$).

For instance $K = \mathbb{R}_+^m$.

Definition

A solution $\bar{x} \in \Omega$ is called *efficient* if there is no $x \in \Omega$ with $f(x) \in \{f(\bar{x})\} - K \setminus \{0\}$.

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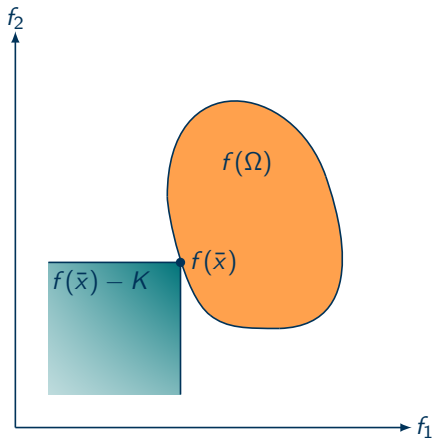
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$$\bar{x} \text{ is efficient} \quad \iff \quad (\{f(\bar{x})\} - K) \cap f(\Omega) = \{f(\bar{x})\}$$

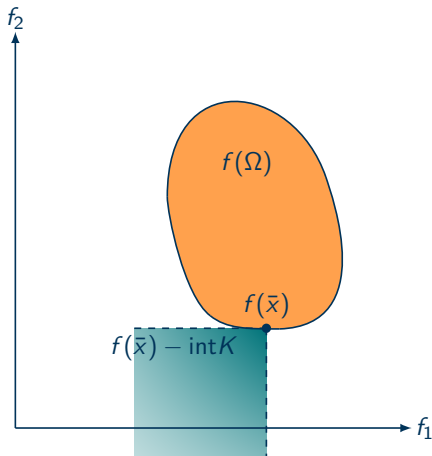
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Efficient solutions



$$\bar{x} \text{ is efficient} \iff (\{f(\bar{x})\} - K) \cap f(\Omega) = \{f(\bar{x})\}$$

Weakly efficient solutions



\bar{x} is weakly efficient $\iff (\{f(\bar{x})\} - \text{int}(K)) \cap f(\Omega) = \emptyset$

Multi-objective decision robustness

Recall: \bar{x} is efficient if there is no $x \in \Omega$ with

$$f(x) \in \{f(\bar{x})\} - K \setminus \{0\}.$$

We replace $f(x)$ by $f_Z(x) = \{f(x+z) \in \mathbb{R}^m \mid z \in Z\}$:

Definition

An element $\bar{x} \in \Omega$ is called *decision robust efficient* with respect to Z , if $\bar{x} \in X$ and if there is no $x \in X$ with

$$f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}.$$

Multi-objective decision robustness

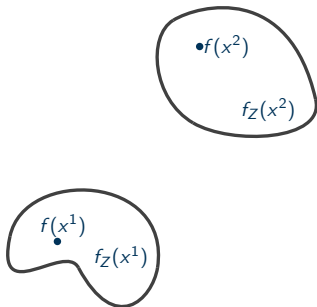
Here: $m = 2$ and $K = \mathbb{R}_+^2$. Let $x^1, x^2 \in X$:

• $f(x^2)$

$f(x^1)$

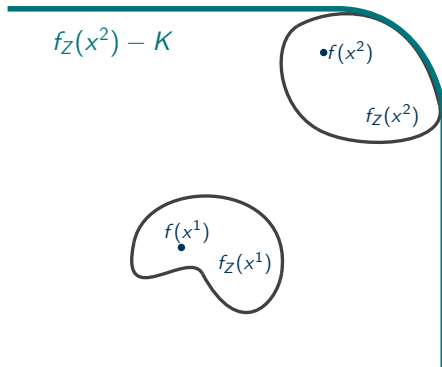
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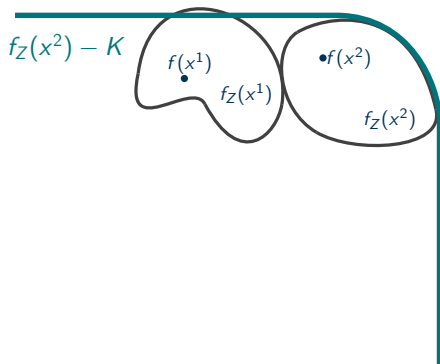
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x^2 is not decision robust efficient with respect to Z as
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Weak multi-objective decision robustness

Recall: \bar{x} is weakly efficient if there is no $x \in \Omega$ with

$$f(x) \in \{f(\bar{x})\} - \text{int}(K).$$

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Example:

Let $m = 2$ and $f = id$, $\Omega = B(0, 3)$ and $Z = B(0, 1)$.

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$$\bar{x} \in \{x \in \partial X \mid x_1 \leq 0, x_2 \leq 0\}$$

are decision robust efficient w.r.t. Z .

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Note: the set of efficient solutions of $\min_{x \in X} f(x)$ is $\{x \in \partial X \mid x_1 \leq 0, x_2 \leq 0\}$.

Relation to the scalar-valued case

Lemma

For $m = 1$, $K = \mathbb{R}_+$ we have

$$\bar{x} \in \min_{x \in X} \sup_{z \in Z} f(x + z) \Leftrightarrow \bar{x} \text{ decision robust (weakly) efficient.}$$

Other Robustness (minmax) approaches

- ▶ Avikad, Branke 2008, Ehrgott, Ide, Schöbel 2014: Minmax robustness for multi-objective optimization problems, **compare sets** $\{f(x, z) \mid z \in Z\}$

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$$\min_{x \in X} (\max_z f_1(x + z), \dots, \max_z f_m(x + z))$$

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- ▶ Deb, Gupta 2006: minimize **mean value** of function values of δ -neighborhood of x

Decision Robust Efficiency

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- ▶ (Def. :) there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.
- ▶ $\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem

$$\min_{x \in X} f_Z(x).$$

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$$\min_{x \in X} \text{Sup} (f_Z(x)).$$

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$$\min_{x \in X} \text{Sup}(f_Z(x)).$$

Thus we are doing set optimization here!



Section 3

Relation to Set Optimization

Set Optimization Problem

Y real linear space, e.g. $Y = \mathbb{R}^m$, partially ordered by

$K \subseteq Y$ a pointed convex cone, e.g. $K = \mathbb{R}_+^m$,

Ω feasible set,

$F: \Omega \rightrightarrows Y$ set-valued map.

A set optimization problem is an optimization problem of the form

$$\min_{x \in \Omega} F(x). \quad (\text{SOP})$$

Set Optimization - Vector Approach

$$\min_{x \in \Omega} F(x)$$

(SOP)

Set Optimization - Vector Approach

$$\min_{x \in \Omega} F(x) \quad (\text{SOP})$$

Optimality using the vector approach:

Definition

A pair (\bar{x}, \bar{y}) with $\bar{x} \in \Omega$ and $\bar{y} \in F(\bar{x})$ is called a minimizer of (SOP) if

$$(\{\bar{y}\} - K) \cap F(\Omega) = \{\bar{y}\},$$

where

$$F(\Omega) = \bigcup_{x \in \Omega} F(x),$$

i.e. if \bar{y} is a minimal element of $F(\Omega)$.

Set Optimization - Vector Approach

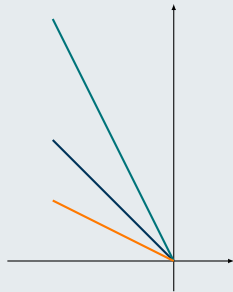
Example:

Let $\Omega = \{\frac{1}{2}, 1, 2\}$, $K = \mathbb{R}_+^2$, and

$$F(2) = \{(-t, 2t) \in \mathbb{R}^2 \mid t \in [0, 2]\}.$$

$$F(1) = \{(-t, t) \in \mathbb{R}^2 \mid t \in [0, 2]\},$$

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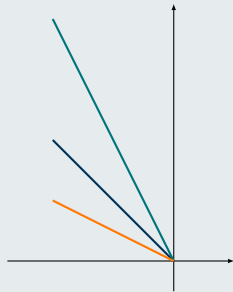
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Then for instance all pairs (\bar{x}, \bar{y}) with $\bar{y} = (0, 0)$ and $\bar{x} \in \Omega$ are minimizers of $\min_{x \in \Omega} F(x)$.

Set Optimization - Vector Approach

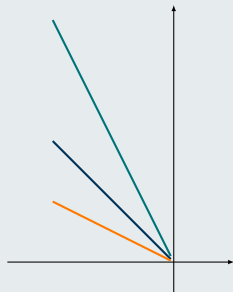
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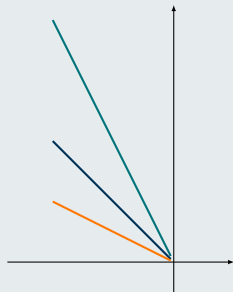
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Set Optimization - Vector Approach

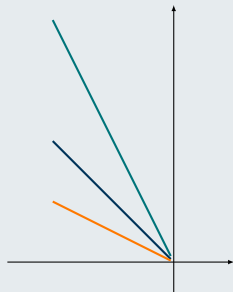
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Then only pairs (\bar{x}, \bar{y}) with $\bar{y} \in F(\frac{1}{2})$ and $\bar{x} = \frac{1}{2}$ are minimizers of $\min_{x \in \Omega} F(x)$.

In general only one element \bar{y} does not imply that the whole set $F(\bar{x})$ is in a certain sense minimal with respect to all sets $F(x)$ with $x \in \Omega$!

Set Optimization - Set Approach

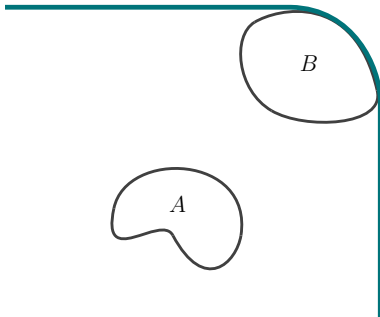
We need order relations for comparing sets! Let $A, B \subseteq \mathbb{R}^m$.

Set Optimization - Set Approach

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- ▶ u -type less order relation \preceq_u is defined by

$$A \preceq_u B \Leftrightarrow A \subseteq B - K.$$



Relation to Set Optimization: Set Relations



$$\begin{aligned} A \preceq_u B & \quad :\Leftrightarrow \quad A \subseteq B - K. \\ & \quad \Leftrightarrow \quad (\forall a \in A \exists b \in B : a \leq b) \end{aligned}$$

... is reflexive and transitive

Relation to Set Optimization: Set Relations



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$$\begin{aligned} A \preceq_{K \setminus \{0\}} B & :\Leftrightarrow A \subseteq B - K \setminus \{0\}. \\ & \Leftrightarrow (\forall a \in A \exists b \in B : a \leq b) \end{aligned}$$



$$\begin{aligned} A \preceq_{\text{int}(K)} B & :\Leftrightarrow A \subseteq B - \text{int}(K). \\ & \Leftrightarrow (\forall a \in A \exists b \in B : a < b) \end{aligned}$$

Relation to Set Optimization: Set Relations



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Using these set relations we can replace for instance

$f_Z(x) \subseteq f_Z(\bar{x}) - K$ in the definition of multi-objective decision robust efficiency by $f_Z(x) \preceq_u f_Z(\bar{x})$.

More Set Relations

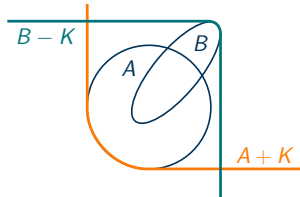
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- ▶ l -type less order relation \preccurlyeq_l

$$\begin{aligned} A \preccurlyeq_l B & :\Leftrightarrow B \subseteq A + K. \\ & \Leftrightarrow (\forall b \in B \exists a \in A : a \leq b) \end{aligned}$$

- ▶ set less order relation: $A \preccurlyeq_s B :\Leftrightarrow A \preccurlyeq_u B \wedge A \preccurlyeq_l B.$



Optimal Solution in Set Optimization

Definition (Rodríguez-Marín, Sama '07)

Let a nonempty set $\Omega \subseteq \mathbb{R}^n$ and a set-valued map $F: X \rightrightarrows \mathbb{R}^m$ be given with $F(x) \neq \emptyset$ for all $x \in \Omega$.

An element $\bar{x} \in \Omega$ is called a *strictly optimal solution* for the set-valued optimization problem

$$\min_{x \in \Omega} F(x)$$

w.r.t. \preceq if

there exists no $x \in \Omega \setminus \{\bar{x}\}$ with $F(x) \preceq F(\bar{x})$.

Example: Set Approach

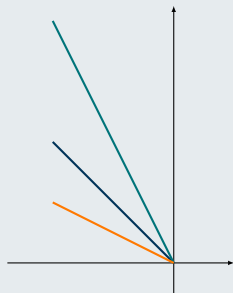
Example:

Let $\Omega = \{\frac{1}{2}, 1, 2\}$, $K = \mathbb{R}_+$, and

$$F(2) = \{(-t, 2t) \in \mathbb{R}^2 \mid t \in [0, 2]\}.$$

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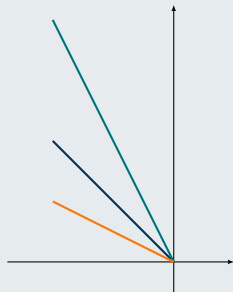
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The only strictly optimal solution is $\bar{x} = \frac{1}{2}$ (for \preceq_u , \preceq_l , \preceq_s).

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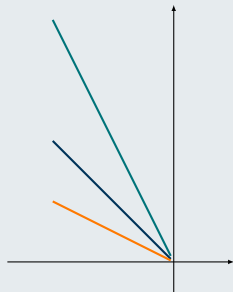
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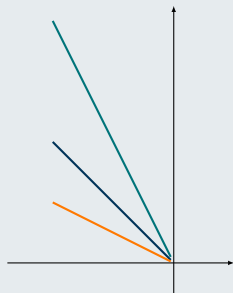
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Decision Robust Efficiency as Set Optimization p.

$$\min_{x \in X} f_Z(x) \quad \text{with} \quad f_Z: X \rightrightarrows \mathbb{R}^m, \quad x \mapsto \{f(x+z) \in \mathbb{R}^m \mid z \in Z\}. \quad (\text{SOP})$$

Corollary

The point $\bar{x} \in X$ is decision robust [weakly/·] efficient w.r.t. Z if and only if $\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem (SOP) w.r.t. $[\preceq_{\text{int}(K)} / \preceq_{K \setminus \{0\}}]$.

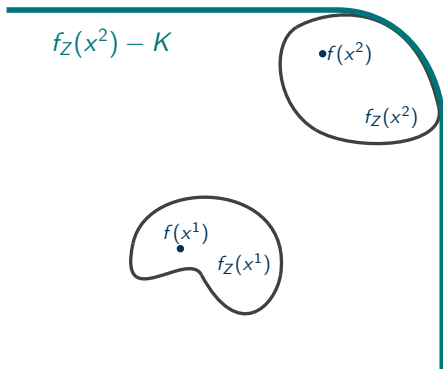
Corollary

An element $\bar{x} \in X$ is decision robust efficient with respect to Z if and only if there is no $x \in X \setminus \{\bar{x}\}$ with

$$\forall z \in Z \exists z' \in Z : f(x+z) \leq f(\bar{x}+z').$$

Multi-objective Decision Robust Efficiency

Here: $m = 2$ and $K = \mathbb{R}_+^2$. Let $x^1, x^2 \in X$:



x^2 is not decision robust efficient with respect to Z as
 $f_Z(x^1) \preceq_{K \setminus \{0\}} f_Z(x^2)$

Decision Robust Efficiency

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- ▶ (Def. :) there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.
- ▶ $\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem

$$\min_{x \in X} f_Z(x).$$

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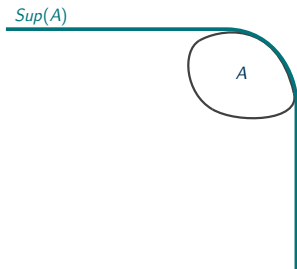
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Supremal Sets

Definition (Nieuwenhuis 1980, Löhne 2011)

Let K be a pointed closed cone with $\text{int}(K) \neq \emptyset$. For a bounded nonempty subset $A \subsetneq \mathbb{R}^m$ the supremal set of A is defined as

$$\text{Sup}(A) = \{y \in \text{cl}(A - K) \mid (\{y\} + \text{int} K) \cap \text{cl}(A - K) = \emptyset\}.$$



If $A \subsetneq \mathbb{R}^m$ is nonempty and bounded, then

$$\text{Sup}(A) \cap A = \text{wMax}(A)$$

where $\text{wMax}(A) = \{y \in A \mid (\{y\} + \text{int}(K)) \cap A = \emptyset\}$.

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By our assumptions we have

$$f_Z(x) - K = \text{cl}(f_Z(x) - K) \notin \{\mathbb{R}^m, \emptyset\}.$$

which implies (see Löhne 2011) that $f_{\text{Sup}}(x) \neq \emptyset$ for all $x \in X$.

Set Relations and Supremal Sets

Lemma

Let $A, B \subseteq \mathbb{R}^m$ be compact sets. Then

$$A - K = \text{Sup}(A) - K \quad \text{and} \quad A - K \setminus \{0\} = \text{Sup}(A) - K \setminus \{0\}$$

and for any $\preceq \in \{\preceq_u, \preceq_{K \setminus \{0\}}, \preceq_{\text{int}(K)}\}$ it holds

$$A \preceq B \quad \Leftrightarrow \quad \text{Sup}(A) \preceq \text{Sup}(B).$$

Decision Robustness by 'min sup'

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- ▶ *there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.*

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Section 4

(Numerical) Approaches to Set Optimization

Stronger Concepts for Necessary Conditions

- ▶ certainly less order relation \preceq_c

$$A \preceq_c B \quad :\Leftrightarrow \quad B \subseteq \bigcap_{a \in A} \{a\} + \mathbb{R}_+^m.$$

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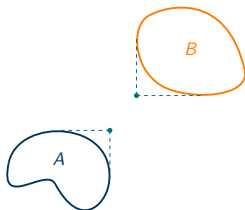
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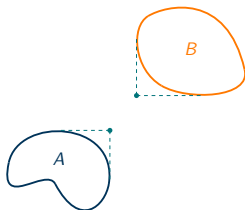
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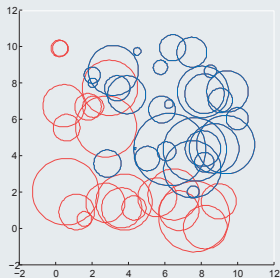


It holds: $A \preceq_c B \Rightarrow A \preceq_s B$.

Necessary Conditions

Example:

Let \mathcal{F} be a family of n disks with random radius $r \in]0, 2[$ and random center points $(x_i, y_i) \in]0, 10[\times]0, 10[$, $i = 1, \dots, n$.



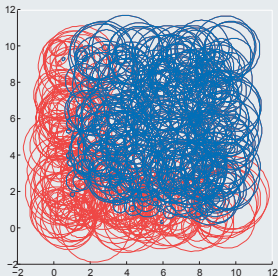
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k	30	66	371	755	3941
$\frac{k}{n}$	0.61	0.66	0.74	0.76	0.79

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Elements from the Dual Cone

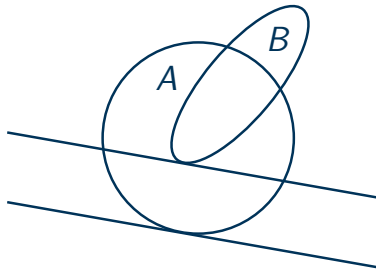
Sets $\emptyset \neq A, B \subseteq \mathbb{R}^m$ compact and convex. Then

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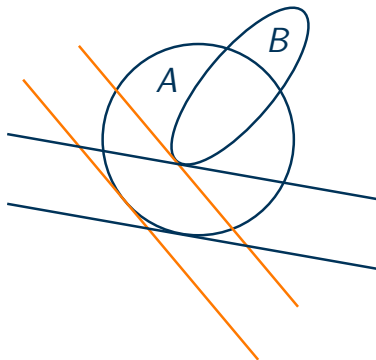
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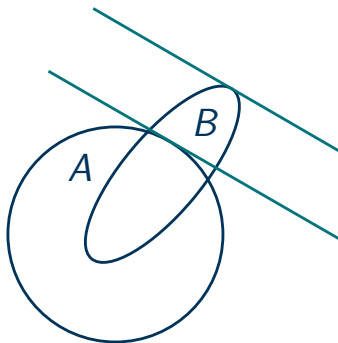
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$$\begin{aligned} A \preceq_s B & :\Leftrightarrow A \preceq_I B \wedge A \preceq_u B \\ & \Leftrightarrow \forall w \in \mathbb{R}_+^m \setminus \{0\} : \min_{a \in A} w^\top a \leq \min_{b \in B} w^\top b \wedge \\ & \quad \max_{a \in A} w^\top a \leq \max_{b \in B} w^\top b \end{aligned}$$

Vectorization

Define for compact and convex sets A

$$v(A)(w) := \begin{pmatrix} \min_{a \in A} w^T a \\ \max_{a \in A} w^T a \end{pmatrix}.$$

Then

$$A \preceq_s B \Leftrightarrow v(A)(w) \leq v(B)(w) \quad \forall w \in \mathbb{R}_+^m \text{ with } \|w\| = 1.$$

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$$\{w^1, \dots, w^l\} \subset \{w \in \mathbb{R}_+^m \mid \|w\| = 1\}.$$

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- ▶ derivative free descent method
- ▶ better results for $F(x)$ polyhedral:
 $F(x) := \{y \in \mathbb{R}^m \mid A(x) \cdot y \leq b(x)\}$ and
 $A(x) \in \mathbb{R}^{p \times m}$, $b(x) \in \mathbb{R}^p$ for all $x \in \Omega$

Nonlinear Scalarization

[Gutiérrez, Jiménez, Miglierina, Molho '15]

Choose $r \in \mathbb{R}_+^m \setminus \{0\}$ and define to sets $\emptyset \neq A, B \subseteq \mathbb{R}^m$:

$$\begin{aligned}g_B(A) &:= \inf\{t \in \mathbb{R} \mid tr + B \subseteq A + \mathbb{R}_+^m\} \\ &= \sup_{b \in B} \inf_{a \in A} (\inf\{t \in \mathbb{R} \mid tr + b \in a + K\}).\end{aligned}$$

Then \bar{x} is a strictly optimal solution for the set-valued optimization problem $\min_{x \in \Omega} F(x)$ if

$$g_{F(\bar{x})}(F(x)) > 0 \text{ for all } x \in \Omega \setminus \{\bar{x}\}.$$

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