Bound set based branch-and-cut algorithms for bi-objective combinatorial optimization problems

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Outline

- Notation and definitions
- Branch and cut
- Bound sets
- Cutting plane algorithm
- Pruning
- Bound set update
- Branching
- Conclusions

Notation

We want to solve a BOCO of the following form

$$\begin{aligned} &\min Cx \\ \text{s.t.: } &Ax \leq b \\ &x \in \{0,1\}^n \end{aligned}$$

For simplicity

- $\mathcal{X} := \{x \in \{0,1\}^n \ : \ Ax \leq b\}$.
- $\underline{\mathcal{X}} := \{x \in [0,1]^n : Ax \leq b\} \leftarrow$ The LP relaxation

Orderings

Definition

For $z^1, z^2 \in \mathbb{R}^2$ we say that

$$z^1 \leqq z^2 \Leftrightarrow z_i^1 \leq z_i^2, \text{ for } i = 1, 2$$
$$z^1 \leq z^2 \Leftrightarrow z^1 \leqq z^2 \text{ and } z^1 \neq z^2$$
$$z^1 < z^2 \Leftrightarrow z_i^1 < z_i^2, \text{ for } i = 1, 2$$

Cones

Definition

By $\mathbb{R}^2_{\geqq}$ we define the set

$$\mathbb{R}^2_{\geqq} = \{z \in \mathbb{R}^2 \ : \ z \geqq 0\}$$

Similarly for $R^2_{\geq}$ and $\mathbb{R}^2_{>}$.

Efficiency

Definition

$\hat{x} \in \mathcal{X}$ is called *efficient* if there does not exist another $x \in \mathcal{X}$ such that

$$Cx \leq C\hat{x}$$

The corresponding outcome vector, $\hat{z} := C\hat{x}$, is called *non–dominated*.

Notation

$\mathcal{X}_E$ set of all efficient solutions.

$\mathcal{Z}_N$ $C\mathcal{X}_E = \{z \in \mathbb{R}^2 \; : \; z = Cx, x \in \mathcal{X}_E\}$

$\bar{\mathcal{Z}}$ upper bound set, $\bar{\mathcal{Z}} \subseteq C\mathcal{X}$.

L lower bound set.

η an active branching node.

$\mathcal{X}(\eta)$ feasible set of node η.

$\underline{\mathcal{X}}(\eta)$ LP relaxed version of $\mathcal{X}(\eta)$.

Branch & cut

Single objective branch and bound:

1. Pick an active node
2. If node is infeasible $\rightarrow$ prune it.
3. Add cuts if necessary
4. Solve relaxation.
   - If solution is integral, update incumbent and prune.
   - If subproblem contains no improving solutions, prune. Else branch.

Bi–objective branch and bound:

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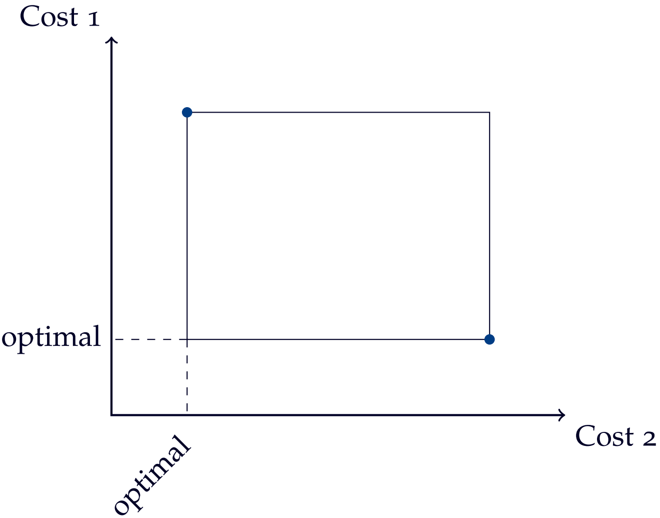

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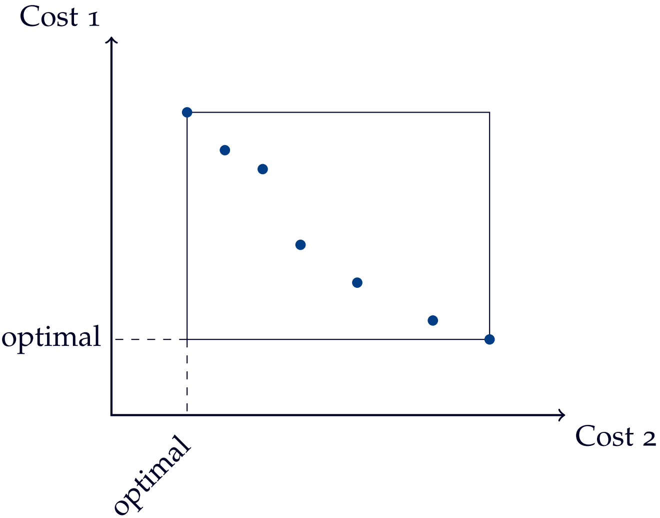

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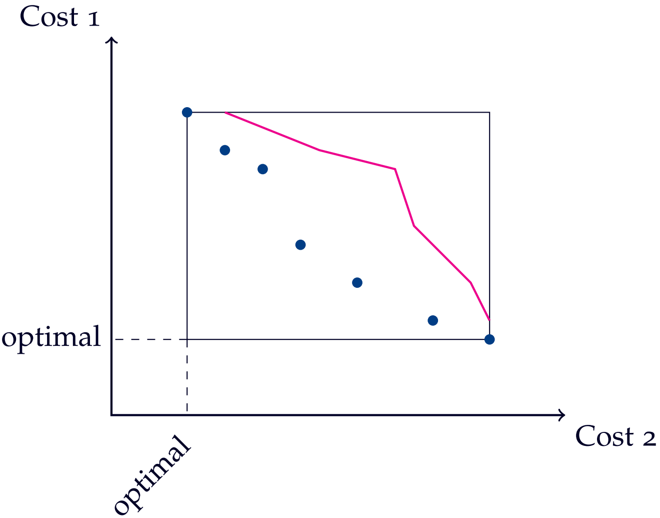

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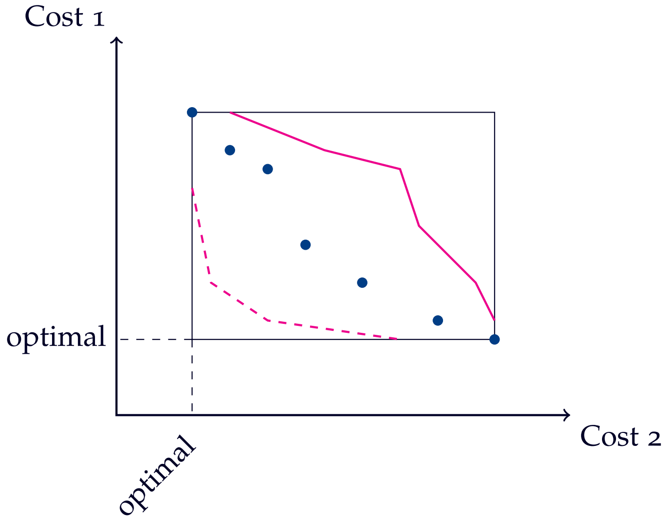

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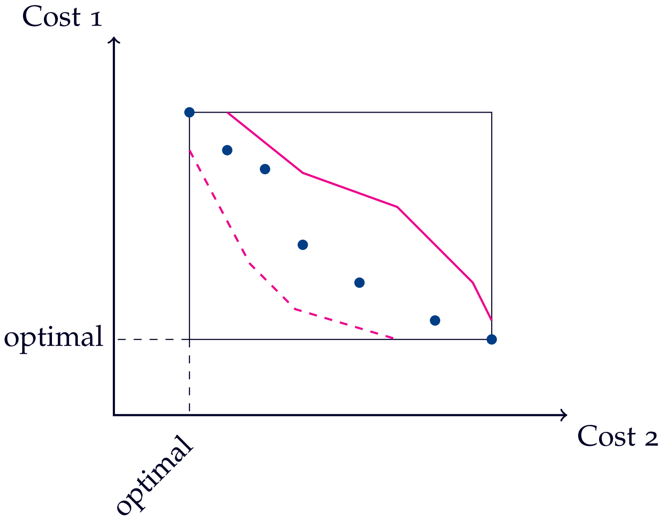

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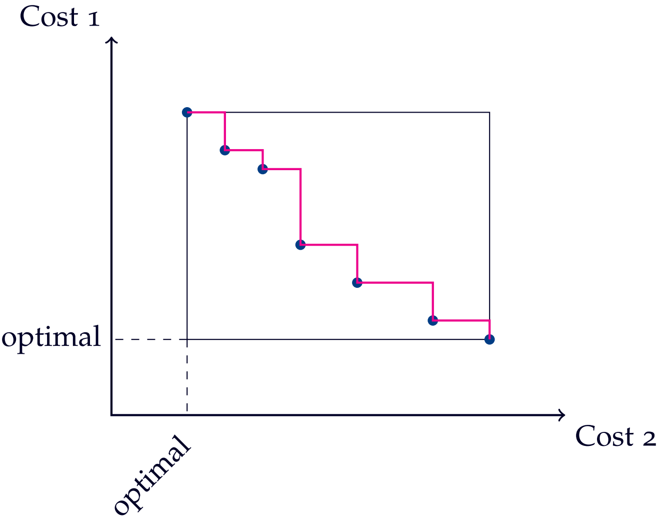

Bound sets

Lower bound set

- Lower bound on the non–dominated frontier

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We only need to search *between* the upper and lower bound sets

Cutting planes

Single objective

- Strengthen the lower bound
- Approximate the integer hull of solutions in the direction of the objective function

Multi objective

- Strengthen the lower bound *set*
- Approximate the integer hull of solutions in the direction of the objective function*s*

Modified NISE–algorithm

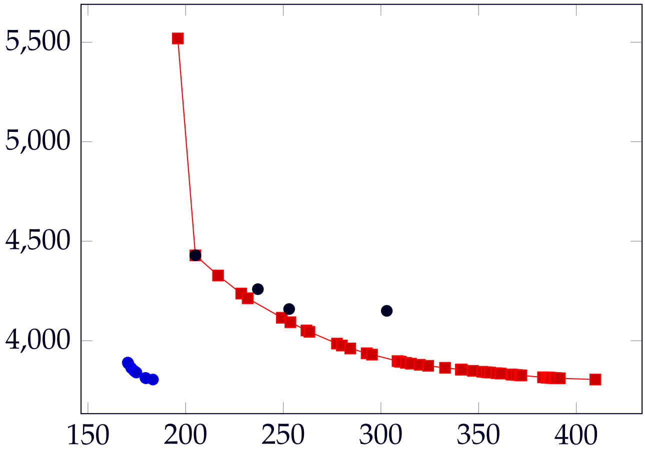
The NISE algorithm works by solving a series of problems of the form

$$\begin{aligned} \min\ & \lambda c^1 x + (1-\lambda) c^2 x \\ \text{s.t.: } & Ax \leqq b \\ & 0 \leqq x \leqq 1 \end{aligned}$$

Modified NISE–algorithm

1. Update λ according to the NISE scheme
2. Solve the weighted sum LP and obtain optimal solution x^*
3. If a cut $\pi^T x \leqq \pi_0$ exists add it, and go back to 2.
4. Else record $c^1 x^*$ and $c^2 x^*$, and go to 1.

An example of cut effect



Bound fathoming

- We assume $\bar{\mathcal{Z}}$ is initialized with the lexicographic minimizers.

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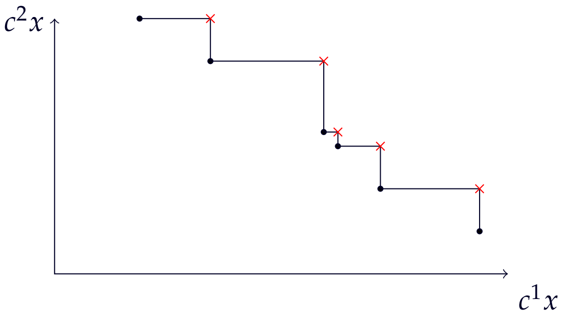
Theorem

A subproblem corresponding to branching node η contains no efficient solutions, if the set

$$L(\eta) + \mathbb{R}^2_{\geqq}$$

contains no local Nadir points.

Bound fathoming – An illustration



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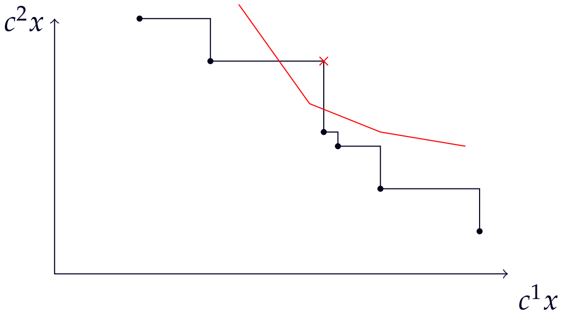

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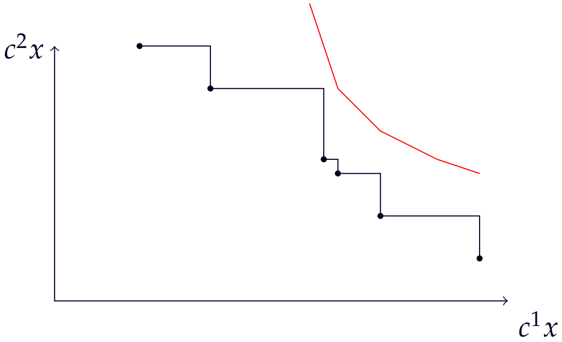

Bound fathoming: Explicit PIP–test

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  - Get the extreme points of the frontier

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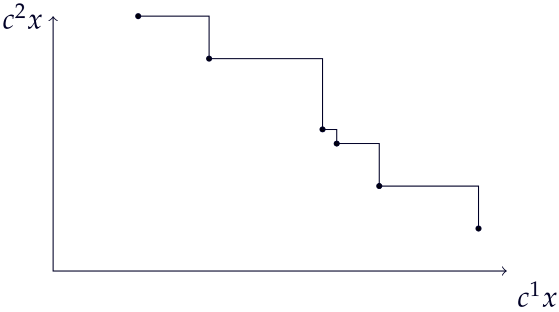
- Solve the Bi–objective LP-relaxation of the node using NISE algorithm
  - Get the extreme points of the frontier
- Intersect with the bounding box from lex-min solutions

Bound fathoming: Explicit PIP–test

How to check this?

- Solve the Bi–objective LP-relaxation of the node using NISE algorithm
  - Get the extreme points of the frontier
- Intersect with the bounding box from lex-min solutions
- Use a PIP algorithm

Bound fathoming: Explicit PIP–test



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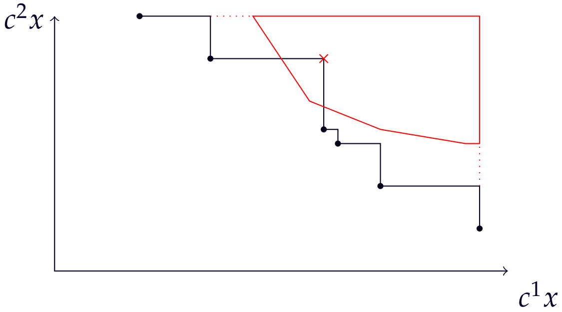


Bound fathoming: Explicit LP–test

Perform the test using linear programming.

- z^N is a local Nadir point
- $\{\underline{z}^1, \ldots, \underline{z}^L\}$ extreme points of $(C\underline{\mathcal{X}}(\eta))_N$.

$$\min s_1 + s_2 \tag{1}$$

$$\text{s.t.:} \sum_{l=1}^{L} \underline{z}_1^l \lambda_l - s_1 \leq z_1^N, \tag{2}$$

$$\sum_{l=1}^{L} \underline{z}_2^l \lambda_l - s_2 \leq z_2^N, \tag{3}$$

$$\sum_{l=1}^{L} \lambda_l = 1. \qquad s_1, s_2 \geq 0. \tag{4}$$

Implicit LP–test

Implicit test using linear programming

- z^N is a local Nadir point
- $\underline{\mathcal{X}}$ LP relaxation

$$\begin{aligned} \min\ & s_1 + s_2 \\ \text{s.t.: } & c^1 x \leqq z_1^N, \\ & c^2 x \leqq z_2^N, \\ & x \in \underline{\mathcal{X}}. \\ & s_1, s_2 \geq 0. \end{aligned}$$

Nodes are *not* dominated

Simple test to check if a node is *not* dominated:

Theorem

A branching node η *cannot be pruned by previous theorem, if there exits* $\lambda \in (0,1)$ *and* $z \in \mathcal{N}(\bar{\mathcal{Z}})$ *such that*

$$Cx^{\lambda} \leq z$$

where $x^{\lambda} \in \arg\min\{(\lambda c^1 + (1-\lambda)c^2)x \ : \ x \in \underline{\mathcal{X}}(\eta)\}$.

Bound set updating

- Solve scalarized LP–relaxation

$$\min\{(\lambda c^1 + (1-\lambda)c^2)x \ : \ x \in \underline{\mathcal{X}}(\eta)\}$$

before solving bi–objective LP–relaxation.

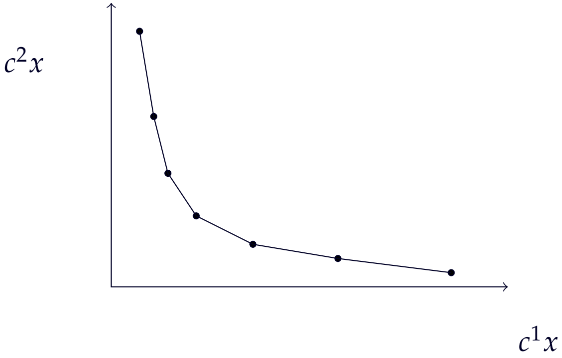
Bound set updating

- Solve scalarized LP–relaxation

$$\min\{(\lambda c^1 + (1-\lambda)c^2)x \ : \ x \in \underline{\mathcal{X}}(\eta)\}$$

  before solving bi–objective LP–relaxation.
- Inherit lower bound set of parent node, and update!

Bound set updating – Illustration



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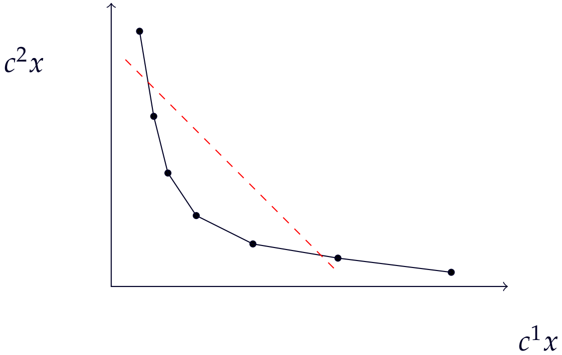

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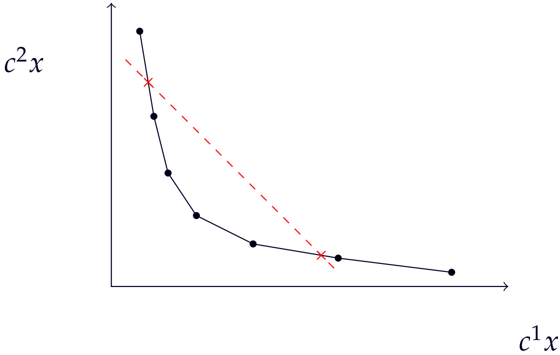

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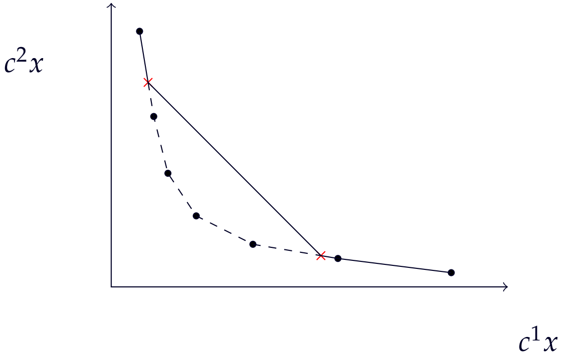
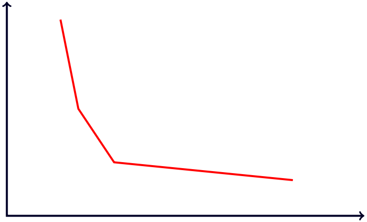


When should we update/resolve

- If we branch in objective space, child nodes should be resolved (more on branching in a minute).
- If the lower bound set from scalarization strictly dominates that of parent node, then we resolve



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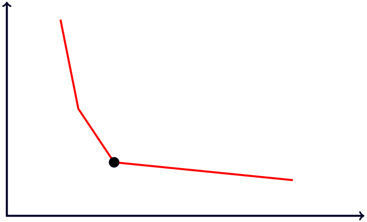

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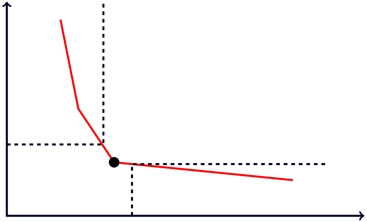

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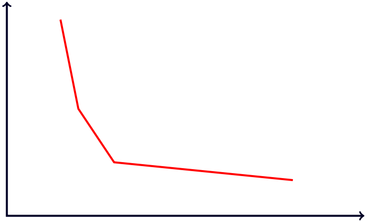

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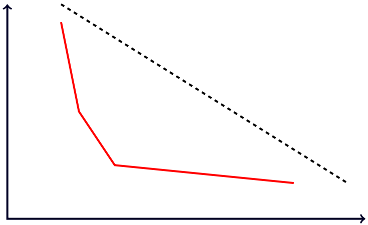

When should we branch?

Single objective branch and bound:

- Pick an active node
- If node is infeasible → prune it.
- Solve relaxation.
  - If solution is integral, update incumbent and prune.
  - If lower bound is worse than incumbent, prune. Else branch.

Bi–objective branch and bound

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  - If solution(s) integral, update incumbent *set* and **branch**.
  - If subproblem contains no efficient solutions, prune. Else **branch**.

Integer branching – No–good inequalities

Let $\bar{x} \in \mathcal{X}$. Create one! child node with the inequality

$$\sum_{i:\bar{x}_i=1} (1 - x_i) + \sum_{i:\bar{x}_i=0} x_i \geq 1$$

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- Might be many *equivalent solutions*
- Use Pareto branching!

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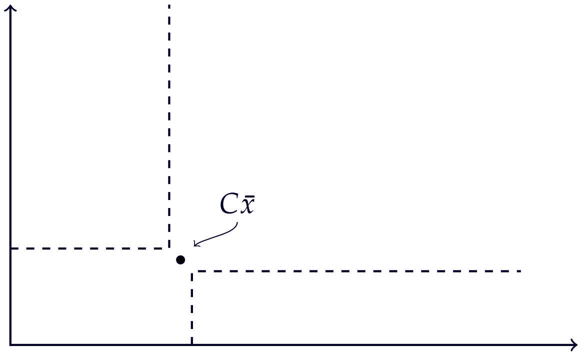

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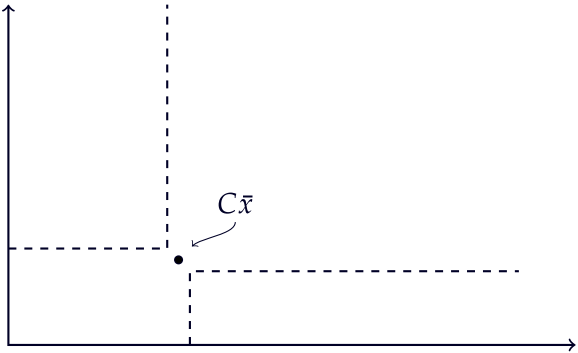

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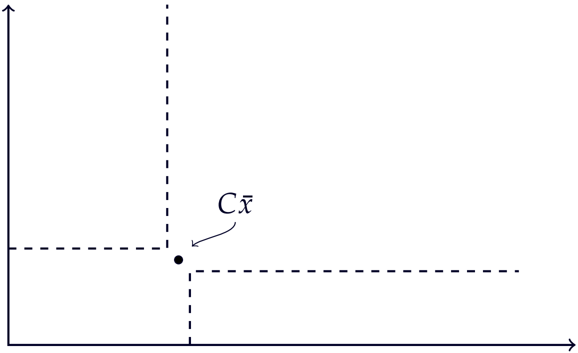

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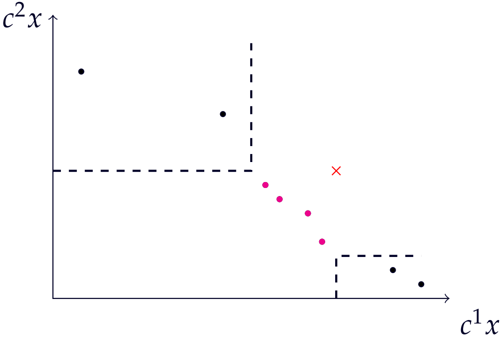
- Generalize to *Pareto branching*

Pareto branching – Illustration



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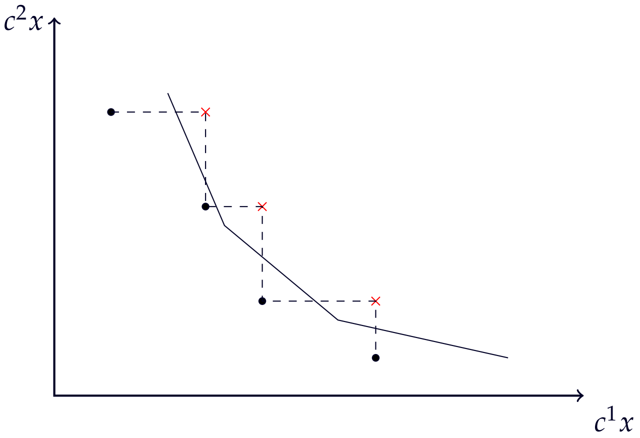
- All non–dominated out comes in the sub–problem η maps to

$$\bigcup_{z \in \mathcal{N}^L(\eta)} \left(\{z\} - \mathbb{R}^2_{\geqq} \right)$$

Extended Pareto branching – Illustration



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  1. When stating the lower bound sets explicitly, LP based test worse than point–in–polytope test.
  2. When stating the lower bound sets implicitly, extended Pareto branching is not improving the performance

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  - It does! The algorithm becomes much more robust and also faster.
- Tested an updating strategy of the lower bound set
  - Works very well. Lower bounds are worse, but we can check many more subproblems.

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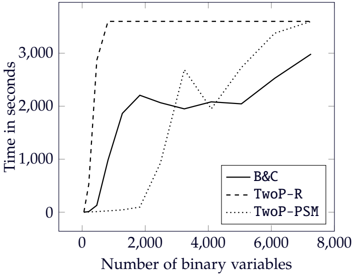

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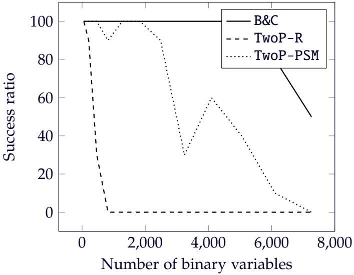

Questions
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