Set-based minmax robust efficiency for uncertain multi-objective optimization

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joint work with
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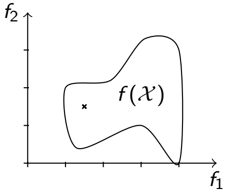
Definition (Multi-objective optimization problem)

Given a feasible set $\mathcal{X} \subset \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^k$, a multi-objective optimization problem is given by

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in \mathcal{X} \end{array}$$

In multi-objective optimization one searches for the set of *nondominated* points $f(\overline{x})$ with $\overline{x} \in \mathcal{X}$, i.e., where there is no $x' \in \mathcal{X} \setminus \{\overline{x}\}$ such that $f_i(x') \leq f_i(\overline{x})$ for all $i = 1, \ldots, k$.
The according solution $\overline{x}$ is called *efficient*.



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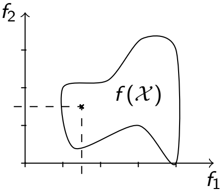

Uncertainties

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Uncertain optimization problem $\mathcal{P}(\mathcal{U})$: Family of (deterministic) optimization problems

$$\begin{array}{lll} \mathcal{P}(\xi) & \min & f(x,\xi) \\ & s.t. & x \in \mathcal{X}, \end{array}$$

where $\xi \in \mathcal{U}$.

The question arises:

When to call a solution to this family of optimization problems *robust optimal*?

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Different concepts of robustness for single objective optimization problems

- Minmax robustness (Soyster, 1973, Ben-Tal & Nemirovski, 1998):
  $\min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} f(x, \xi)$

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Different concepts of robustness for single objective optimization problems

- Minmax robustness (Soyster, 1973, Ben-Tal & Nemirovski, 1998):
  $\min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} f(x, \xi)$
- many more (e.g., Ben-Tal et al., 2009, Goerigk & Schöbel, 2013)

Robust
Optimization

Robust Optimization

Multi-Objective Optimization

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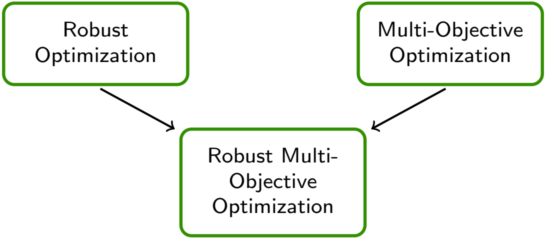
Multi-Objective Optimization

- Both robust and multi-objective optimization important in research and real-world applications



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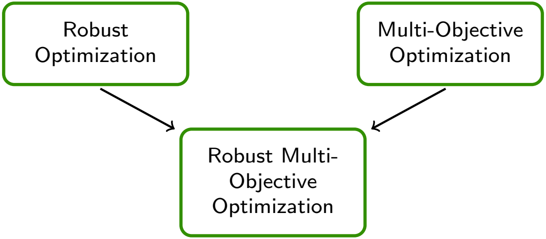

- Both robust and multi-objective optimization important in research and real-world applications
- Connection of these two topics quite new (e.g., Kuroiwa & Lee, 2012; Witting, 2012)
- Some other works available as pre-prints (e.g., Doolittle et al., 2012; Kuhn et al., 2012)

Definition (Uncertain multi-objective problem)

Given an uncertainty set $\mathcal{U}$, a feasible set $\mathcal{X} \subset \mathbb{R}^n$ and a function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^k$, an uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$ is given by the family of all problems

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The question arises

When do we call a solution $x \in \mathcal{X}$ *robust* efficient?

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What is a worst case for the uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$ given by

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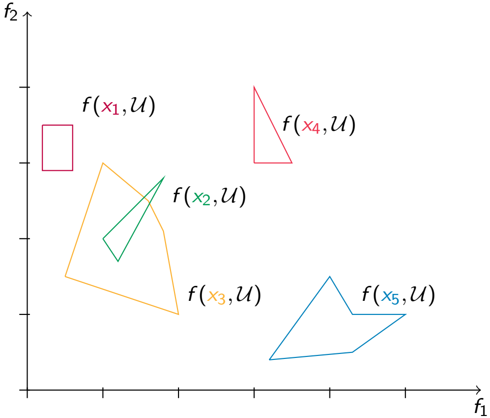
Interpreting the supremum as a *set*



Which of these solutions do we call minmax robust efficient?

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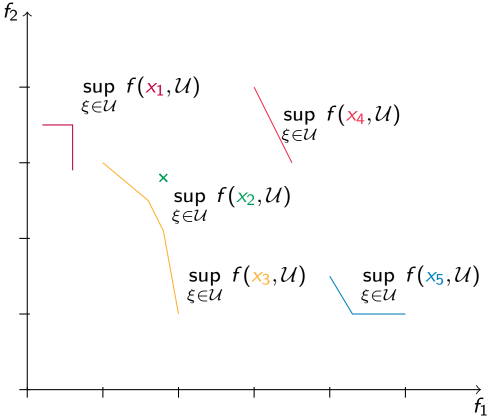

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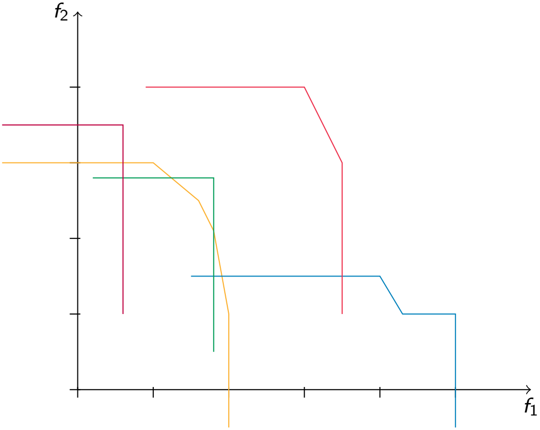

We will call those $x \in \mathcal{X}$ minmax robust efficient, where $f(x,\mathcal{U})$ is *nondominated*.

Definition (Robust efficiency)

Given an uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$ *we call a solution* $\overline{x} \in \mathcal{X}$ minmax robust efficient,

if there is no $x' \in \mathcal{X} \setminus \{\overline{x}\}$ *such that*

$$f(x', \mathcal{U}) \subseteq f(\overline{x}, \mathcal{U}) - \mathbb{R}^k_{\geq}$$

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Definition (Robust efficiency)

Given an uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$ *we call a solution* $\overline{x} \in \mathcal{X}$ minmax robust weakly efficient,

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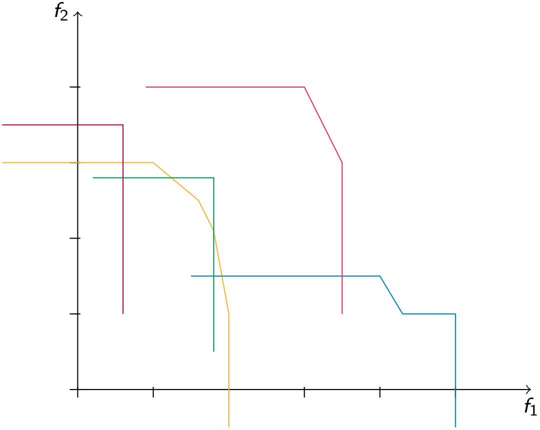
Interpreting the supremum as a *set*



The orange, blue, green and purple solutions are minmax robust strictly efficient, the red one is not even minmax robust weakly efficient.

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- How to calculate robust efficient solutions?

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Question

- How to calculate robust efficient solutions?
  - First idea: Find solutions by solving a robust single-objective problem

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- For $|\mathcal{U}| = 1$ these minmax robust efficiency definitions reduce to the definition of efficiency
- For $k = 1$ the definition of minmax robust weakly efficiency reduces to the definition of minmax robust optimality

Question

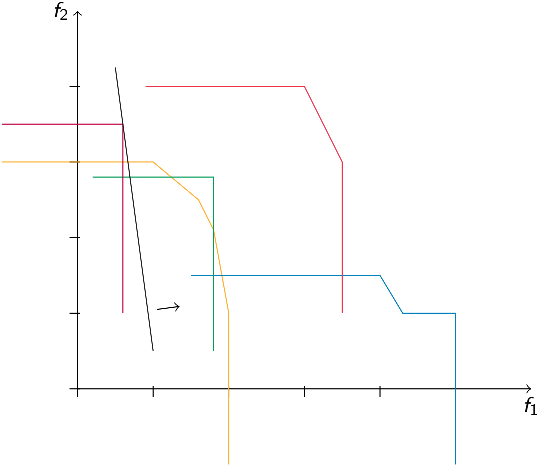
- How to calculate robust efficient solutions?
  - First idea: Find solutions by solving a robust single-objective problem
  - Second idea: Find solutions by solving a deterministic multi-objective problem

Theorem

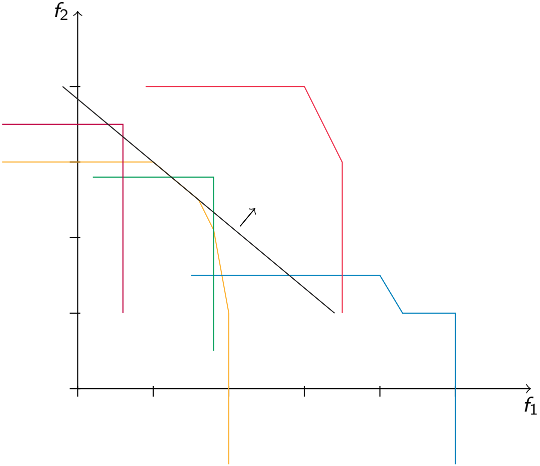
If $\overline{x} \in \mathcal{X}$ is the unique minimizer of

$$\sup_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$$

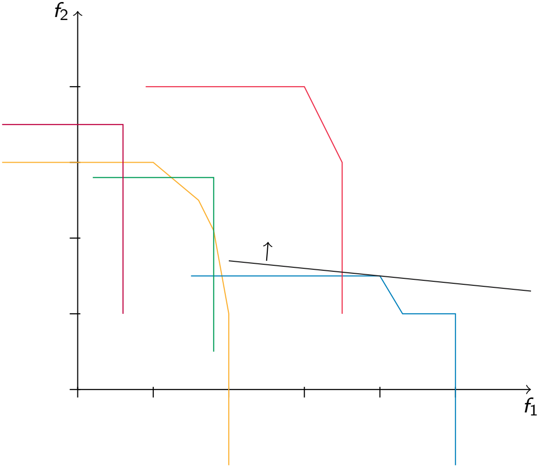
over $\mathcal{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, $\overline{x}$ is minmax robust strictly efficient.



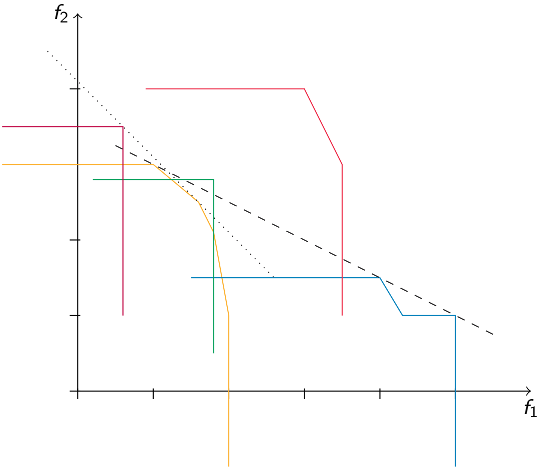
The purple solution is minmax robust strictly efficient.



The orange solution is minmax robust strictly efficient.



The blue solution is minmax robust strictly efficient.



The green minmax robust strictly efficient solution is no optimal solution for any scalarization problem.

Theorem

If $\overline{x} \in \mathcal{X}$ is the unique minimizer of

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over $\mathcal{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, $\overline{x}$ is minmax robust strictly efficient.

Theorem

If $\max\limits_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$ exists for all $x \in \mathcal{X}$ and $\overline{x} \in \mathcal{X}$ is a minimizer of

$$\max_{\xi \in \mathcal{U}} \sum_{i=1}^{k} \lambda_i f_i(x, \xi)$$

over $\mathcal{X}$ for some $\lambda \in \mathbb{R}^k_{\geq}$, then $\overline{x}$ is minmax robust weakly efficient.

Definition

$$
\begin{array}{lll}
\epsilon\mathcal{C}_{\mathcal{P}(\mathcal{U})}(\epsilon, i) & \min & \sup\limits_{\xi \in \mathcal{U}} f_i(x, \xi) \\
& \text{s.t.} & f_j(x, \xi) \leq \epsilon_j \ \forall\, j \neq i, \ \forall\, \xi \in \mathcal{U} \\
& & x \in \mathcal{X}
\end{array}
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Theorem

Given a problem $\mathcal{P}(\mathcal{U})$.

a) *If $\overline{x} \in \mathcal{X}$ is the unique optimal solution to $\epsilon\mathcal{C}_{\mathcal{P}(\mathcal{U})}(\epsilon, i)$ for some i, then it is minmax robust strictly efficient.*

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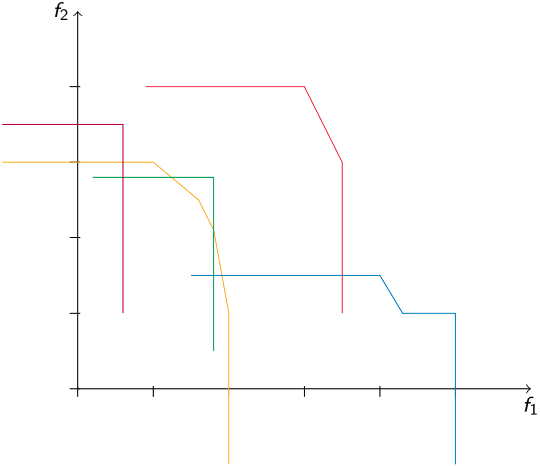
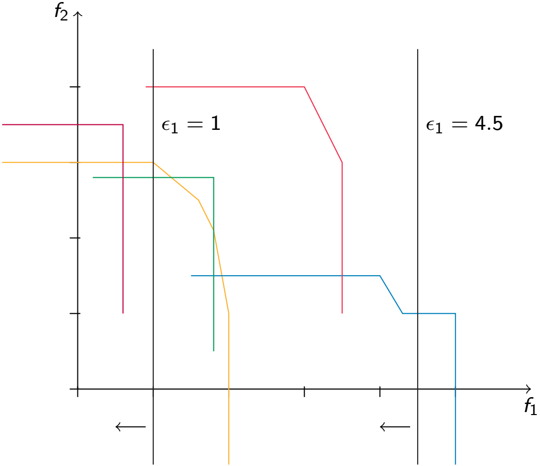
Given a problem $\mathcal{P}(\mathcal{U})$.

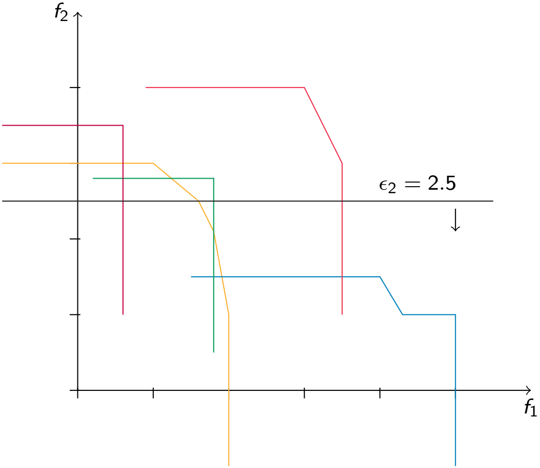
a) *If $\overline{x} \in \mathcal{X}$ is the unique optimal solution to $\epsilon\mathcal{C}_{\mathcal{P}(\mathcal{U})}(\epsilon, i)$ for some i, then it is minmax robust strictly efficient.*

b) *If $\overline{x} \in \mathcal{X}$ is an optimal solution to $\epsilon\mathcal{C}_{\mathcal{P}(\mathcal{U})}(\epsilon, i)$ for some i and $\max\limits_{\xi \in \mathcal{U}} f_i(x, \xi)$ exists for all $x \in \mathcal{X}$, then $\overline{x}$ is minmax robust weakly efficient.*


f_2
f_1



The green solution minimizes $f_2(x,\xi)$ over $\{x \in \mathcal{X}: \ f_1(x,\xi) \leq 4.5 \ \forall \xi \in \mathcal{U}\}$,
the purple solution minimizes $f_2(x,\xi)$ over $\{x \in \mathcal{X}: \ f_1(x,\xi) \leq 1 \ \forall \xi \in \mathcal{U}\}$,



The blue solution minimizes $f_1(x, \xi)$ over $\{x \in \mathcal{X} : \ f_2(x, \xi) \leq 2.5 \ \forall \xi \in \mathcal{U}\}$
The orange solution cannot be found with the ϵ-constraint method.

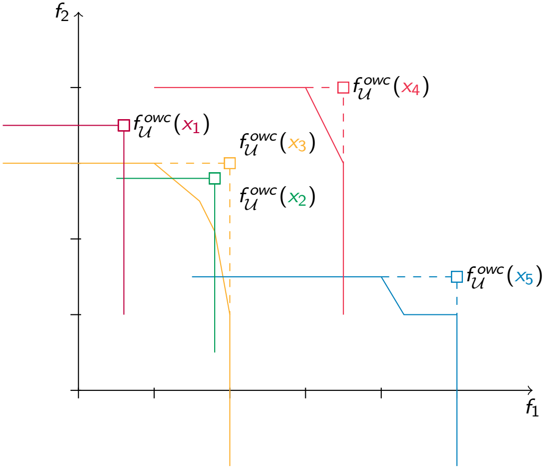
Definition

We formulate a new problem

$$\mathcal{OWC} \min_{x \in \mathbb{X}} f_{\mathcal{U}}^{owc}(x)$$

where

$$f_{\mathcal{U}}^{owc}(x) := \begin{pmatrix} \sup\limits_{\xi \in \mathcal{U}} f_1(x,\xi) \\ \sup\limits_{\xi \in \mathcal{U}} f_2(x,\xi) \\ \vdots \\ \sup\limits_{\xi \in \mathcal{U}} f_k(x,\xi) \end{pmatrix}$$


f_2
$f^{owc}_{\mathcal{U}}(x_4)$
$f^{owc}_{\mathcal{U}}(x_1)$
$f^{owc}_{\mathcal{U}}(x_3)$
$f^{owc}_{\mathcal{U}}(x_2)$
$f^{owc}_{\mathcal{U}}(x_5)$
f_1

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(1) *If $\overline{x} \in \mathbb{X}$ is a strictly efficient solution for $(\mathcal{OWC})$, then it is minmax robust strictly efficient for $\mathcal{P}(\mathcal{U})$.*

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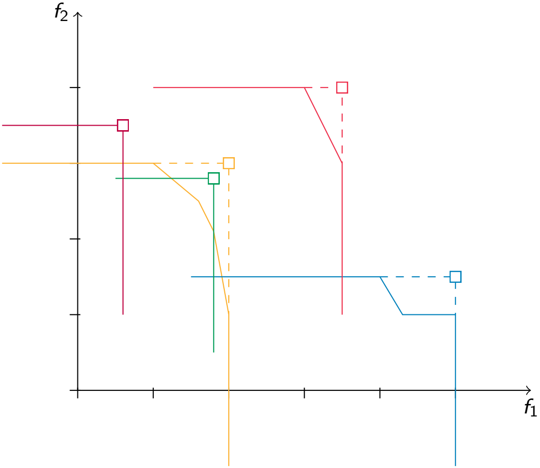
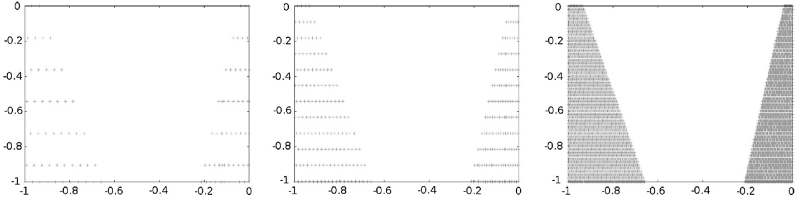
(2) *If $\max_{\xi \in \mathcal{U}} f_i(x, \xi)$ exists for all $i = 1, \ldots, k$ and $x \in \mathbb{X}$ and $\overline{x}$ is weakly efficient for $(\mathcal{OWC})$, it is minmax robust weakly efficient for $\mathcal{P}(\mathcal{U})$.*



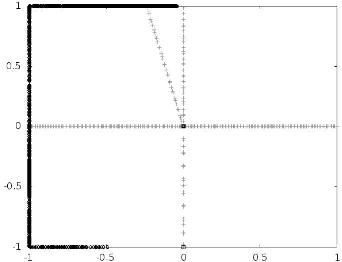
The strictly efficient solutions of $(\mathcal{OWC})$ are the purple, green and blue solutions.
The orange solution cannot be found this way.

ϵ-constraint method



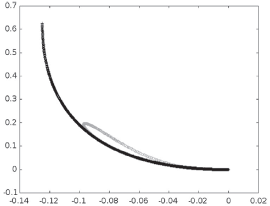
Approximation of the minmax robust efficient set of the ϵ-constraint method (left to right: $\epsilon = 1,\ 0.5,\ 0.1$)

weighted sum and ϵ-constraint scalarization

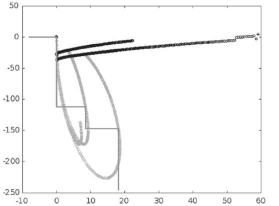


Minmax robust efficient solutions obtained by weighted sum (black) and ϵ-constraint (grey) scalarization (left linear, right quadratic)

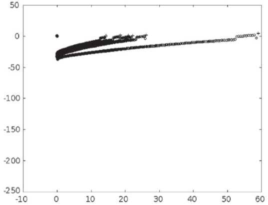
ϵ-constraint method



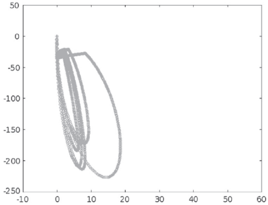
(a) Non-dominated set (black) and objective vectors of the robust efficient solutions (grey) in the nominal scenario.



(b) Objective vectors of the nominal (black) and the robust (grey) efficient solutions in the respective worst cases.



(c) Objective vectors of the nominally efficient solutions under all scenarios.



(d) Objective vectors of the robust efficient solutions under all scenarios.

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Thank you for your attention!