A linear bound on the number of scalarizations needed to solve discrete tricriteria optimization problems

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Recent advances in multi-objective optimization, Vienna

Outline

1. Introduction
2. Generic decomposition of the 'search region'
3. Redundancy for problems with more than two criteria ($m > 2$)
4. Improved decomposition for $m = 3$
5. Linear bound on the number of subproblems for $m = 3$
6. Conclusion

Notation

General multicriteria problem:

$$\begin{aligned} \min \quad & f(x) = [f_1(x), \ldots, f_m(x)]^\top \\ \text{s.t.} \quad & x \in X \end{aligned}$$

Representation in outcome space:

$$\begin{aligned} \min \quad & z = [z_1, \ldots, z_m]^\top \\ \text{s.t.} \quad & z \in Z \end{aligned}$$

Z discrete, finite set

Goal: Determine the entire nondominated set Z_N

Approach based on scalarizations

- Solve sequence of scalarizations with different parameter choices (‘subproblems’)
- Basic algorithm

Determine initial search region; $s = 1$;
while there is some unexplored region **do**
    Choose region, solve subproblem ‘therein’
    if subproblem infeasible **then**
        remove explored region
    else
        save new nondominated point z^s;
        update search region based on z^s;
    end if
    $s := s + 1$;
end while
Output: Set of nondominated points Z_N

Bound on number of subproblems

Bicriteria case:

- $\mathcal{O}(N)$, at most $2N - 1$, see Aneja & Nair (1979), Chalmet et al. (1986), Ralphs et al. (2006)

Multicriteria case:

- $\mathcal{O}(N^{m-1})$, see Laumanns et al. (2006), Özlen and Azizoğlu (2009), Lokman and Köksalan (2013), Ozlen et al. (2014), Kirlik and Sayın (2014)
  $\Rightarrow \mathcal{O}(N^2)$ for $m = 3$

We show $\mathcal{O}(N)$ for $m = 3$:

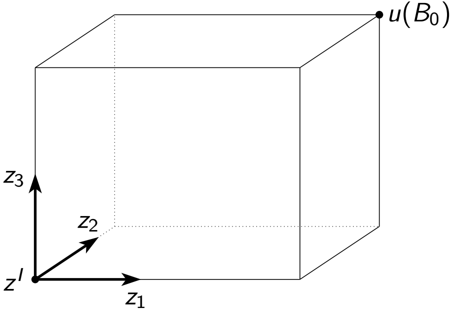
- $3N - 2$ for arbitrary scalarization
- $2N - 1$ for ε-constraint method

Decomposition of search region for $m \geq 2$

Initial search region (box)

$$B_0 := \{z \in \mathbb{R}^m : z^I \leqq z < u\}$$

with $u_i := \max_{x \in X}\{f_i(x)\} + \delta,\ i = 1, \ldots, m,\ \delta > 0$



Decomposition of search region for $m \geq 2$

Initial search region (box)

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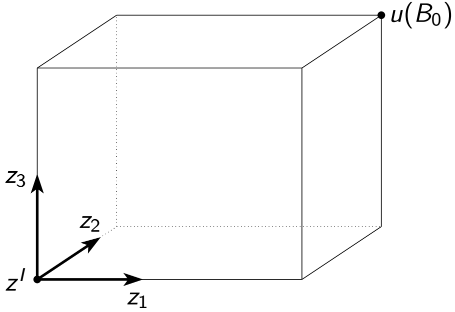
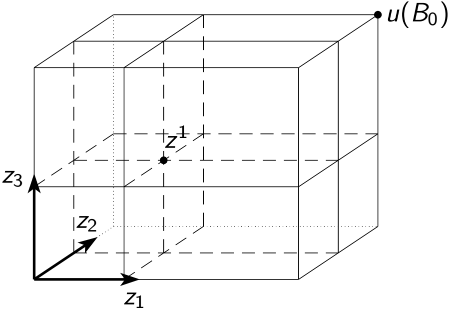
$\rightsquigarrow$ Note: Every box B characterized by $u(B)$



Decomposition of search region for $m \geq 2$

Solve subproblem in $B_0 \rightsquigarrow z^1 \in Z_N \cap B_0$

Insertion of z^1 into B_0

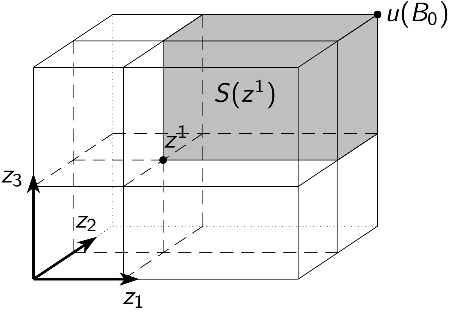


Decomposition of search region for $m \geq 2$

By definition of nondominance:

$$Z_N \cap S(z^1) = \{z^1\} \quad \text{mit} \quad S(z^1) := \{z \in B_0 : z \geqq z^1\}$$

$\Rightarrow$ All $z \in Z_N \setminus \{z^1\}$ contained in $B_0 \setminus S(z^1)$



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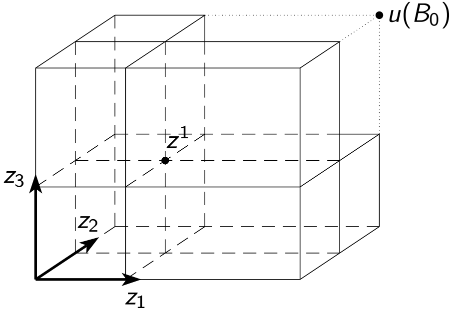
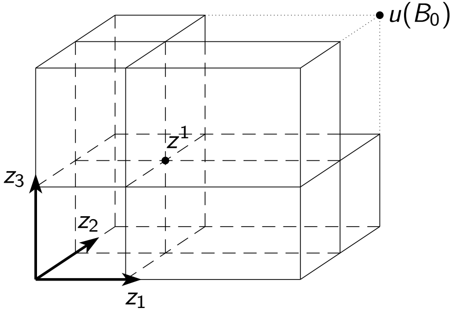


Decomposition of search region for $m \geq 2$

Representation of $B_0 \backslash S(z^1)$ by $\bigcup_{i=1}^{m} B_{1,i}$ with

$$B_{1,i} := \{z \in B_0 : z_i < z_i^1\}, \quad i = 1, \ldots, m,$$

i.e. $u_i(B_{1,i}) := z_i^1$, $u_j(B_{1,i}) := u_j(B_0) \ \forall j \neq i$



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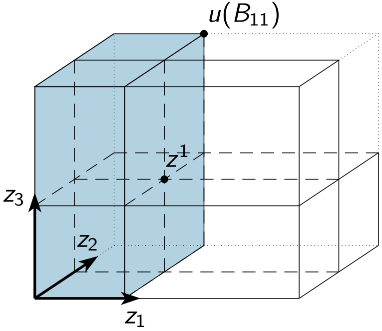

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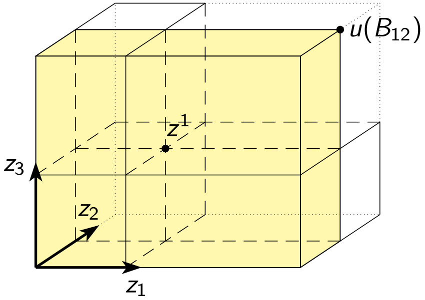

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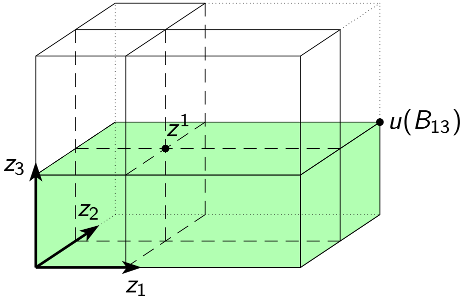
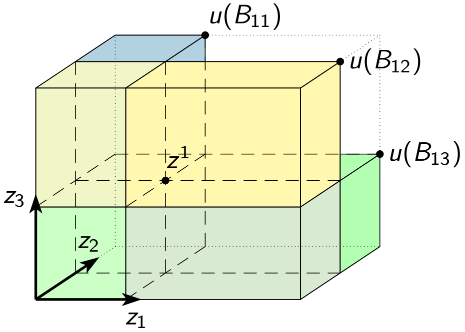


Decomposition of search region for $m \geq 2$

Representation of $B_0 \backslash S(z^1)$ by $\bigcup_{i=1}^{m} B_{1,i}$ with

$$B_{1,i} := \{z \in B_0 : z_i < z_i^1\}, \quad i = 1, \ldots, m,$$

$\Rightarrow$ Decomposition of $B_0 \backslash S(z^1)$ into m (non-disjoint) subboxes



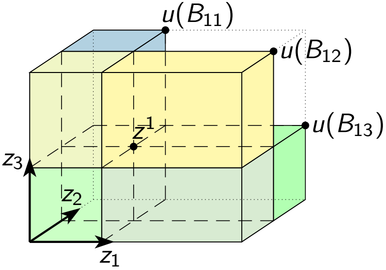
Full m-split

Full m-split

Replace box B wrt. $z^s \in B \cap Z_N$ into m new boxes

$$B_{s,i} := \{z \in B \,:\, z_i < z_i^s\} \quad \forall\, i = 1, \ldots, m$$

(see Dhaenens et al. (2010), Przybylski et al. (2010))



Note: No decomposition into $2^m - 1$ disjoint boxes (cf. Tenfelde-Podehl (2003)) in order to keep number of subproblems small

Full m-split

1. Using full m-split in iterative algorithm:
   In every iteration s, we split every box, in which current nondominated point z^s is contained, into m new boxes.

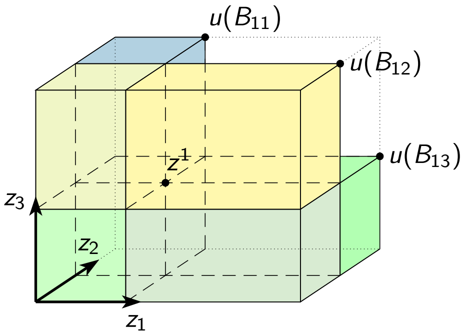
2. Correctness:
   In iteration s, only

   $$S_2(z^s) := \{z \in B_0 : z \geqq z^s\}$$

   is excluded from search region

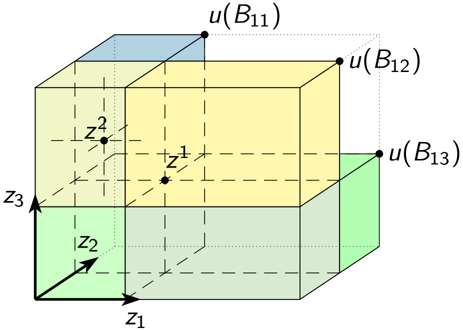
3. Disadvantage of full m-split: Redundant boxes appear for $m \geq 3$

Full m-split: Redundancy for $m \geq 3$



$$\mathcal{B}_2 = \{B_{11}, B_{12}, B_{13}\}$$

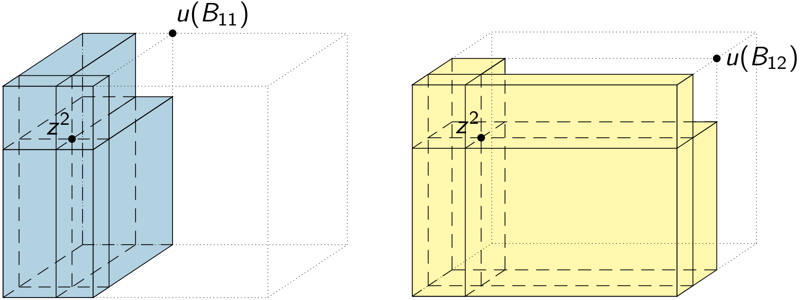
Full m-split: Redundancy for $m \geq 3$



Let $z^2 \in (B_{11} \cap B_{12})$

$\Rightarrow$ Split B_{11} and B_{12} into 3 new boxes, resp.

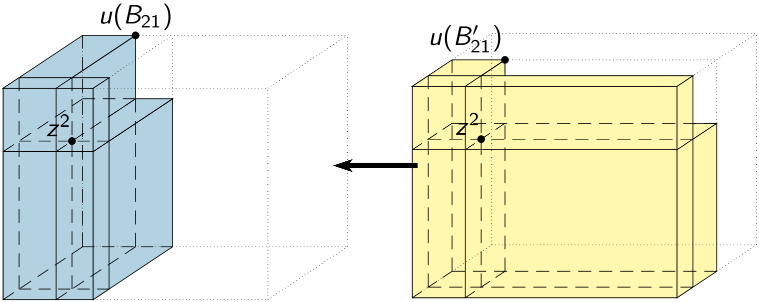
Full m-split: Redundancy for $m \geq 3$



B_{21}, B_{22}, B_{23} $B'_{21}, B'_{22}, B'_{23}$

Full 3-split of B_{11} and B_{12} wrt. z^2

Full m-split: Redundancy for $m \geq 3$

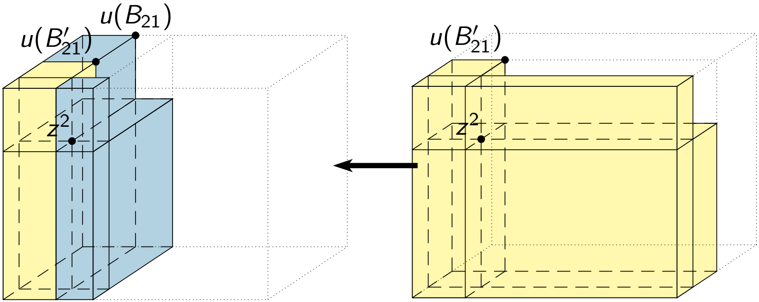


$\underline{\mathbf{B_{21}}}, B_{22}, B_{23}$ $\underline{\mathbf{B'_{21}}}, B'_{22}, B'_{23}$

Split wrt. $i = 1$:

$B'_{21} \subseteq B_{21} \iff u(B'_{21}) \leqq u(B_{21})$

Full m-split: Redundancy for $m \geq 3$



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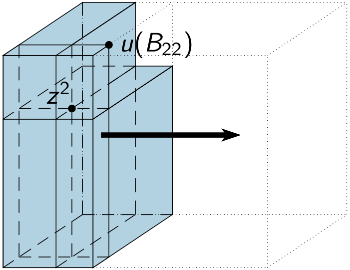
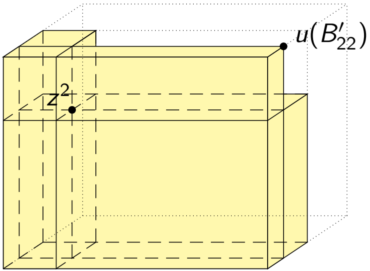


B_{21}, $\underline{\mathbf{B_{22}}}$, B_{23}

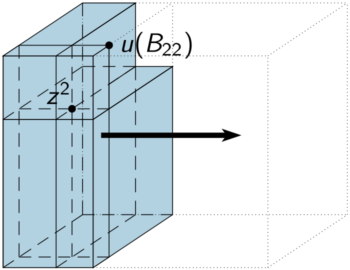


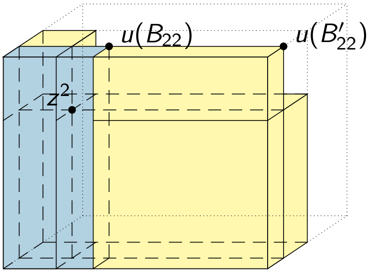
B'_{21}, $\underline{\mathbf{B'_{22}}}$, B'_{23}

Split wrt. $i = 2$:

$$B_{22} \subseteq B'_{22} \iff u(B_{22}) \leqq u(B'_{22})$$

Full m-split: Redundancy for $m \geq 3$



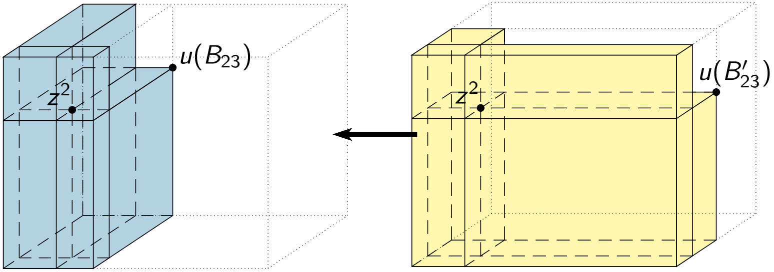


$B_{21}, \underline{\mathbf{B_{22}}}, B_{23}$ $B'_{21}, \underline{\mathbf{B'_{22}}}, B'_{23}$

Split wrt. $i = 2$:

$$B_{22} \subseteq B'_{22} \iff u(B_{22}) \leqq u(B'_{22})$$

Full m-split: Redundancy for $m \geq 3$

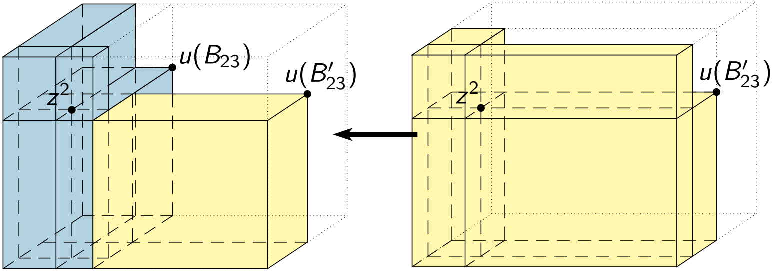


$B_{21}, B_{22}, \underline{\mathbf{B_{23}}}$ $B'_{21}, B'_{22}, \underline{\mathbf{B'_{23}}}$

Split wrt. $i = 3$:

no redundancy

Full m-split: Redundancy for $m \geq 3$



$B_{21}, B_{22}, \underline{\mathbf{B_{23}}}$

$B'_{21}, B'_{22}, \underline{\mathbf{B'_{23}}}$

Split wrt. $i = 3$:
no redundancy

Redundancy

Full m-split produces redundant boxes

- Example: already in 2nd iteration, two of the six new boxes redundant
- if redundant boxes are kept in decomposition
  - additional, unnecessary subproblems are solved
  - running time of algorithm increased

$\Rightarrow$ avoid generation of redundant boxes

Identifying redundant boxes:

1. compare upper bounds $u(B)$ pairwise, remove redundant ones (Przybylski et al. (2010))
2. detect redundant boxes before their creation, i.e. only split box if resulting box non-redundant

Individual subsets

Observation:

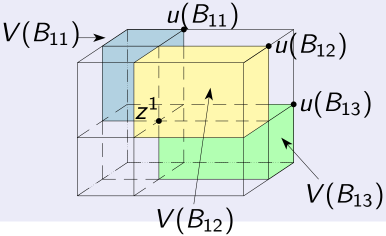
B non-redundant $\iff$ B contains non-empty subset which is not part of any other box of the decomposition

Definition (Individual subsets)

For every $\bar{B} \in \mathcal{B}_s$, the set

$$V(\bar{B}) := \bar{B} \setminus \left(\bigcup_{B \in \mathcal{B}_s \setminus \{\bar{B}\}} B \right)$$

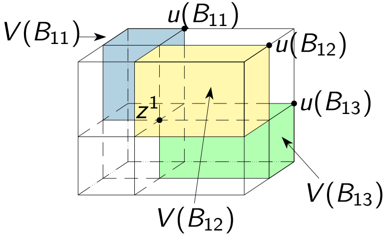
is called individual subset of $\bar{B}$.



Individual subsets

- Maintaining non-redundant boxes $\iff$ maintaining boxes with non-empty individual subset
- Split box B wrt. $i \iff V(B_i) \neq \emptyset$
- Goal: Explicit representation of $V(B)$

Simplifying assumption:
General position: For all $z, \bar{z} \in Z_N, z \neq \bar{z}$, let $z_i \neq \bar{z}_i$ for all $i = 1, \ldots, m$.



Representation of $V(B)$

- Consider box $\bar{B}$ of (non-redundant) decomposition $\mathcal{B}_s$ and component $i \in \{1, \dots, m\}$.
- A box $\hat{B} \in \mathcal{B}_s$ with

$$u_i(\hat{B}) < u_i(\bar{B})$$
$$u_j(\hat{B}) \geq u_j(\bar{B}) \quad \forall j \neq i$$

and

$$u_i(\hat{B}) = \max\{u_i(B) : B \in \mathcal{B}_s, u_i(B) < u_i(\bar{B})\}$$

limits $V(\bar{B})$ wrt. i.

- We call $\hat{B}$ neighbor of $\bar{B}$ wrt. component i in iteration s (Notation: $\hat{B} = B_i^s(\bar{B})$).

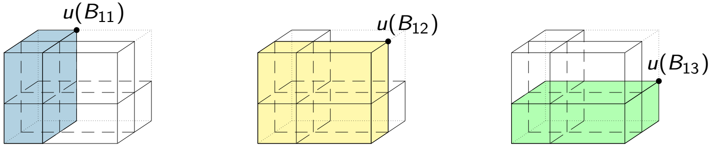
Lemma (Neighbor of a box for $m = 3$)

For every $\bar{B} \in \mathcal{B}_s$ *and every* $i \in \{1, 2, 3\}$ *with* $u_i(\bar{B}) > \min_{B \in \mathcal{B}_s}\{u_i(B)\}$ $\exists!\, \hat{B} \in \mathcal{B}_s$ *such that*

$$u_i(\hat{B}) < u_i(\bar{B})$$
$$u_j(\hat{B}) > u_j(\bar{B}) \quad \text{for some } j \neq i$$
$$u_k(\hat{B}) = u_k(\bar{B}) \quad \text{for } k \neq i, j$$

and

$$u_i(\hat{B}) = \max\{u_i(B) : B \in \mathcal{B}_s, u_i(B) < u_i(\bar{B})\}$$



Representation of $V(B)$ for $m = 3$

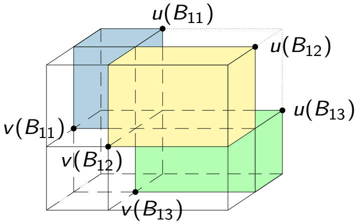
The individual subsets $V(B), B \in \mathcal{B}_s$ can be represented as

$$V(B) = \{z \in B_0 : v(B) \leqq z < u(B)\}$$

with

$$v_i(B) := \begin{cases} u_i(B_i^s(B)), & \text{if } B_i^s(B) \neq \emptyset \\ z_i^I, & \text{otherwise} \end{cases}, \quad i \in \{1, 2, 3\}$$

and $B_i^s(B)$ denoting the neighbor of B wrt. i in iteration s,



e.g.

$$v(B_{12}) = \begin{pmatrix} u_1(B_{11}) \\ z_2^I \\ u_3(B_{13}) \end{pmatrix}$$

The v-split

$$\boxed{\text{Recall: Split box } B \text{ wrt } i \iff V(B_i) \neq \emptyset}$$

Lemma

Let $z^s \in B$, and let B_i be box obtained from B by a split wrt. component $i \in \{1, 2, 3\}$.

$$\textit{Then } B_i \textit{ is non-redundant} \iff z_i^s \geq v_i(B).$$

Definition (v-split)

Let $z^s \in B$. We call split of $B \in \mathcal{B}_s$ wrt. components $i \in \{1, 2, 3\}$, for which

$$z_i^s \geq v_i(B)$$

holds, v-split of B.

The v-split algorithm

Changes in comparison to generic algorithm:

1. Save $v(B)$
2. Apply v-split: For every $B \in \mathcal{B}_s$ with $z^s \in B$ and every $i \in \{1, 2, 3\}$ check whether
$$z_i^s \geq v_i(B)$$
3. Update of $v(B)$:

   For all new $B_i, i \in \{1, 2, 3\}$:
   - sort $u(B_i)'s$ increasingly wrt. $j \neq i$, decreasingly wrt. $k \neq i, j$:
   $$z_j^s < \dots \leq u_j(\hat{B}_i) \leq u_j(\bar{B}_i) \leq u_j(\tilde{B}_i) \leq \dots$$
   $$\dots \geq u_k(\hat{B}_i) \geq u_k(\bar{B}_i) \geq u_k(\tilde{B}_i) \geq \dots > z_k^s$$
   - Set $v_j(\bar{B}_i) := u_j(\hat{B}_i)$, $v_k(\bar{B}_i) := v_k(\tilde{B}_i)$ and so on

The v-split algorithm

Changes in comparison to generic algorithm:

1. Save $v(B)$
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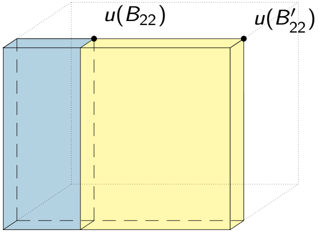
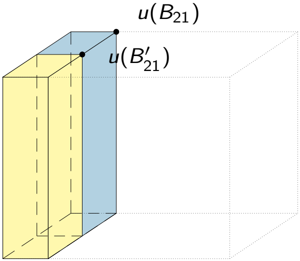
Example revisited

Full m-split: Two redundant boxes

- B_{22} (Split of B_{11} wrt. $i = 2$)
- B'_{21} (Split of B_{12} wrt. $i = 1$)





Example revisited

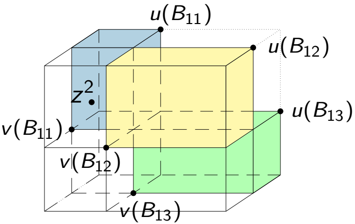
Full *m*-split: Two redundant boxes

- B_{22} (Split of B_{11} wrt. $i = 2$)
- B'_{21} (Split of B_{12} wrt. $i = 1$)

v-Split in B_{11}:

$$\begin{array}{lll} z_1^2 & > & v_1(B_{11}) \quad \checkmark \\ z_2^2 & < & v_2(B_{11}) \\ z_3^2 & > & v_3(B_{11}) \quad \checkmark \end{array}$$

$\Rightarrow$ Split B_{11} wrt. $i = 1$ and $i = 3$
(Redundant box B_{22} not generated!)



Example revisited

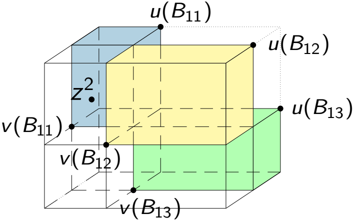
Full m-split: Two redundant boxes

- B_{22} (Split of B_{11} wrt. $i = 2$)
- B'_{21} (Split of B_{12} wrt. $i = 1$)

v-Split in B_{12}:

$$
\begin{array}{lll l}
z_1^2 & < & v_1(B_{12}) & \\
z_2^2 & > & v_2(B_{12}) & \checkmark \\
z_3^2 & > & v_3(B_{12}) & \checkmark
\end{array}
$$

$\Rightarrow$ Split B_{12} wrt. $i = 2$ and $i = 3$
(Redundant box B'_{21} not generated!)



Linear bound on the number of subproblems

Lemma

In every iteration $s \geq 1$ of the v-split algorithm, in which a new nondominated point z^s is found, the number of boxes in the decomposition increases by at most two.

Proof.

Case 1: one box is split $\Rightarrow$ 3 boxes replace one $(-1 + 3 = 2 \checkmark)$

Case 2: more than one box split:

- every box split wrt. at most 2 components
- no pair of boxes split wrt. to the same 2 components
  $\Rightarrow$ at most $2 \cdot 3 - 3 = 3$ additional boxes
- if 3 boxes split wrt. 2 components
  $\Rightarrow$ exists box which is split wrt. no component $(3 - 1 = 2 \checkmark)$

$\square$

Theorem

For Z_N finite ($N = |Z_N|$) and given appropriate initial search region with $lb = z^I$, the v-split algorithm requires at most $3N - 2$ subproblems in order to generate the entire nondominated set.

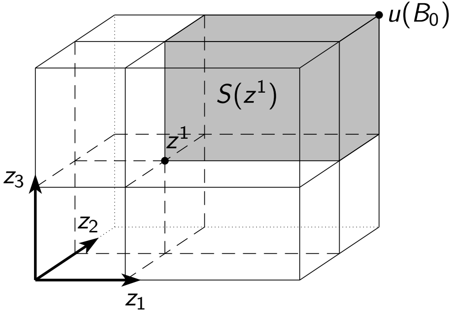
Proof.

- in every iteration one subproblem solved
  $\Rightarrow$ number of subproblems equals number of iterations
- for every nondominated point generated
  $\Rightarrow$ number of boxes increases by at most two (previous Lemma)
- every nondominated point is generated exactly once,
  every empty box is investigated exactly once
  $\Rightarrow$ at most $3N$ boxes explored
- plus initial box $\Rightarrow 3N + 1$
- if $z_i^s = z_i^I$ for $i \in \{1, 2, 3\}$, no box created wrt. $i \Rightarrow 3N - 2$

$\square$

Remarks

1. v-split algorithm and linear bound also valid if nondominated points not in general position
   $\rightsquigarrow$ degenerated individual subsets
2. v-split algorithm independent of particular scalarization
   $\rightsquigarrow$ improved bound $2N - 1$ for ε-constraint method



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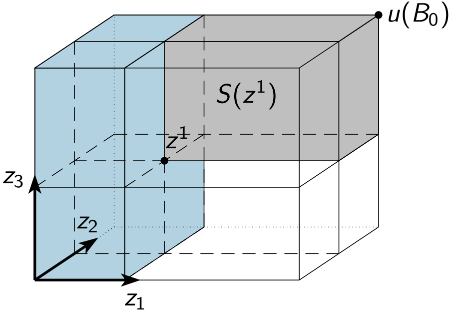

Implementation

Matlab-Implementation of the v-split-algorithm with 3D-visualization

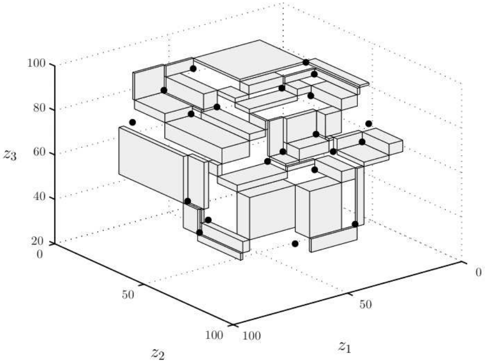

Figure: Example with 21 nondominated points
(Illustration of the individual subsets of all boxes at the end of the algorithm)

Computational experiments

- Multidimensional, tricriteria knapsack problem
- Test problem from Laumanns et al. (2006)
- Original data
- Scalarizations: Weighted Tchebycheff (WT) und ε-Constraint (EC), both in variants Two-stage (TS) und Augmented (A)
- IBM ILOG CPLEX Optimization Studio Version 12.5
- CPLEX called without parallelizing
- MATLAB R2013a
- 4x Intel Xeon E7540 CPUs (2.0 GHz), 128 GB memory

Computational experiments (1)

Validation of theoretical upper bounds $3N - 2$ (WT) and $2N - 1$ (EC)

n	N		WT CPU	WT #SP	EC CPU	EC #SP
10	9	TS	10.03	25	7.97	17
		A	7.81		6.09	
20	61	TS	56.42	181	43.29	121
		A	42.72		30.02	
30	195	TS	213.31	583	163.15	389
		A	163.29		114.39	
40	389	TS	464.47	1165	361.74	777
		A	361.01		257.64	
50	1048	TS	1552.56	3142	1369.89	2095
		A	1174.90		1012.15	

$\Rightarrow$ Bounds met precisely in all instances

Computational experiments (2)

Comparison with algorithms of

1. Lokman & Köksalan (2013)
2. Kirlik & Sayın (2014)
3. Özlen, Burton & MacRae (2014)

Recall:

- $\mathcal{O}(N^2)$ subproblems for $m = 3$
- ε-Constraint Scalarization

$\Rightarrow$ Use (EC) for v-split-algorithm

- Note: All algorithms reimplemented in Matlab and tested with both variants (TS) and (A)

Computational experiments (2)

1. Comparison with Lokman & Köksalan (2013):

n	N		v-Split EC CPU	v-Split EC #SP	LK CPU	LK #SP
10	9	TS	7.97	17	9.48	20
		A	6.09		6.67	
20	61	TS	43.29	121	53.04	127
		A	30.02		31.76	128
30	195	TS	163.15	389	267.88	468
		A	114.39		159.12	464
40	389	TS	361.74	777	657.58	852
		A	257.64		445.07	
50	1048	TS	1369.89	2095	4772.89	2193
		A	1012.15		4129.47	2200

Observation: v-Split-Algorithm requires

- less subproblems,
- less CPU time than LK

Computational experiments (2)

2. Comparison with Kirlik & Sayın (2014):

n	N		v-Split EC CPU	v-Split EC #SP	KS CPU	KS #SP
10	9	TS	7.97	17	8.50	17
		A	6.09		6.07	
20	61	TS	43.29	121	50.08	115
		A	30.02		30.26	
30	195	TS	163.15	389	242.42	373
		A	114.39		155.89	372
40	389	TS	361.74	777	701.95	739
		A	257.64		516.15	738
50	1048	TS	1369.89	2095	4174.48	1913
		A	1012.15		3603.67	1914

Observation: v-split-algorithm requires

- more subproblems in general,
- but less CPU time than KS

Computational experiments (2)

3. Comparison with Özlen, Burton & MacRae (2014):

n	N		v-Split EC CPU	v-Split EC #SP	OBM CPU	OBM #SP
10	9	TS	7.97	17	8.85	19
		A	6.09		6.46	
20	61	TS	43.29	121	48.50	117
		A	30.02		28.83	
30	195	TS	163.15	389	197.05	375
		A	114.39		110.33	374
40	389	TS	361.74	777	430.84	741
		A	257.64		246.68	740
50	1048	TS	1369.89	2095	1533.93	1915
		A	1012.15		945.35	1916

Observation: v-split-algorithm requires

- more subproblems in general,
- partially less, partially more CPU time than OBM

Summary

Conclusion:

1. new split for tricriteria problems avoids redundant boxes
2. linear bound on number of subproblems:
   - $3N - 2$ for arbitrary scalarization
   - $2N - 1$ for ε-constraint method
3. competes with state-of-the-art algorithms

Ongoing research:

1. explicit use of neighborhood structure in algorithm
2. generalization to $m \geq 4$
3. representative subsets / application to continuous problems

Thank you! Questions?