

Branch and bound for biobjective mixed integer optimization

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Biobjective MILP

$$\begin{array}{ll} \min & y_1 = \mathbf{c}_1^\top \mathbf{x} \\ \min & y_2 = \mathbf{c}_2^\top \mathbf{x} \\ \text{s.t.} & \mathbf{x} \in X \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) \end{array}$$

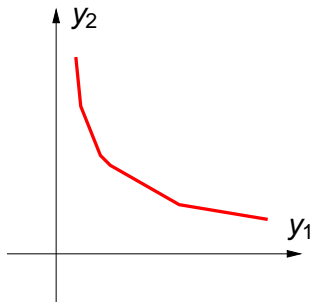
where $X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. Assume X is bounded.

Pareto set of a biobjective linear problem

Consider the problem $\min\{(\mathbf{c}_1^\top \mathbf{x}, \mathbf{c}_2^\top \mathbf{x}) : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.

Y is a union of segments.

The set $Y + \mathbb{R}_+^2$ is convex.

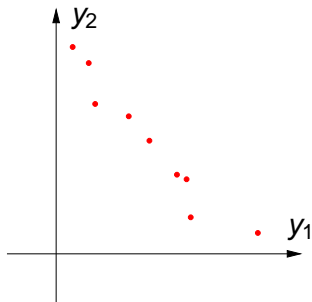


Pareto set of a biobjective pure integer problem

Consider the problem $\min\{(\mathbf{c}_1^\top \mathbf{x}, \mathbf{c}_2^\top \mathbf{x}) : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n\}$.

Y is finite.

The set $Y + \mathbb{R}_+^2$ is **nonconvex**.



Pareto set of a biobjective **pure integer** problem

- ▶ We can generate all (non-dominated) solutions by exhaustive search in finite time.
- ▶ The Pareto set is given by a subset of the leaf nodes of the branch-and-bound tree.
- ▶ Same for the **mixed integer** case if \mathbf{c}_1 or \mathbf{c}_2 are nonzero only in the integer components

Pareto set of a biobjective mixed integer problem

- ▶ Consider \bar{X} as the set of all *fixings* of the integer variables $\mathbf{x}_{1:p} = \bar{\mathbf{x}}$:

$$\bar{\mathbf{x}} \in \bar{X} \subset \mathbb{Z}^p$$

and $\bar{\mathbf{x}}$ is such that $\{\mathbf{x} \in X : \mathbf{x}_{1:p} = \bar{\mathbf{x}}\} \neq \emptyset$.

⇒ We get an LP Pareto set $Y_{\bar{\mathbf{x}}}$ for each $\bar{\mathbf{x}} \in \bar{X}$

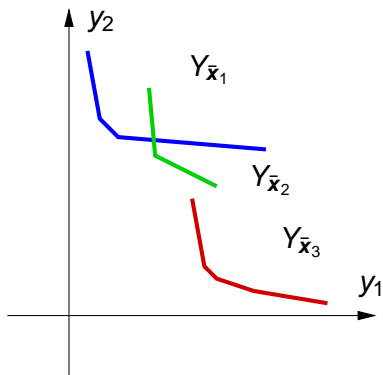
- ▶ Y is a **subset** of the union of all these Pareto sets:

$$Y \subseteq \bigcup_{\bar{\mathbf{x}} \in \bar{X}} Y_{\bar{\mathbf{x}}}$$

(just eliminate its dominated points)

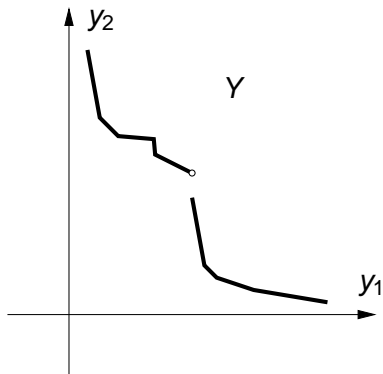
Example

Suppose $|\bar{X}| = 3$, i.e., $\bar{X} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$.



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- ▶ Fathoming rules: Mavrotas & Diakoulakis '98; Ehrgott & Gandibleux '08, Delort & Spanjaard '10
- ▶ TSP: BB where every node is a polynomially solvable BO-ILP (Jozefoviez, Laporte, Semet '11)
- ▶ BOMILP with one pure integer objective function: Costa, Captivo, Climaco '08; Stidsen, Andersen, Dammann '12; Prins, Prodhon, Wolfler Calvo '06
- ▶ Partitioning the objective space and solving multiple MILPs (Savelsbergh & Boland '13)
- ▶ Bi-objective MINLP (D'Ambrosio and Cacchiani, previous talk)

Main idea: **one** run of the branch-and-bound.

- ▶ BB methods for MILP are extremely sophisticated and flexible
- ▶ BB is a tree search with rules to drop entire subtrees
- ▶ Relaxations are refined after branching
- ▶ BB gives estimate how close we are to termination (gap)

Branch-and-bound scheme

procedure BB ($\mathbf{c}_1, \mathbf{c}_2, A, \mathbf{b}$)

$\mathcal{L} \leftarrow \{P\}$

while $\mathcal{L} \neq \emptyset$ **do**

 Select node P_k from \mathcal{L}

if P_k cannot be fathomed **then**

 Solve (P_k)

if P_k not fully searched **then**

 Branch P_k into P'_k and P''_k

$\mathcal{L} \leftarrow (\mathcal{L} \setminus \{P_k\}) \cup \{P'_k, P''_k\}$

Generalizing branch-and-bound from MIPs to BO-MIPs requires:

- ▶ a node solver (no longer an LP \rightarrow parametric simplex method)
- ▶ fathoming rules (no longer “`lower_bound >= cutoff`”)
- ▶ termination criteria (redefine the gap)

Fathoming rules for node k

Rule 1: Fathom if LP relaxation infeasible;

Rule 2: Fathom if its Pareto set Y_k is dominated by the Pareto set Y of the original problem.

However, at node k we know neither Y_k nor Y .

Fathoming rules

Instead of Y :

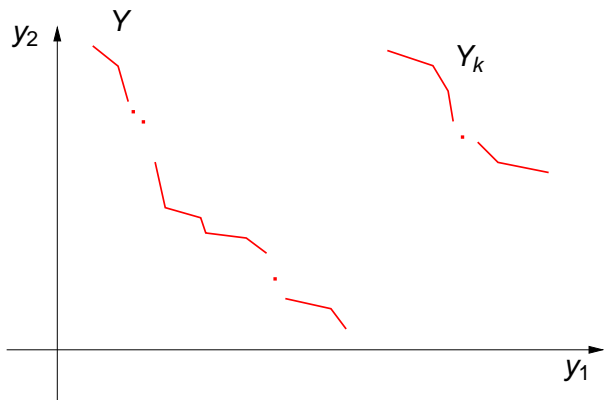
- ▶ take a set Y^{cutoff} of **feasible** solutions that are not dominated by others encountered so far
- ▶ Y^{cutoff} can be updated by adding any integer feasible \mathbf{x}

Instead of Y_k :

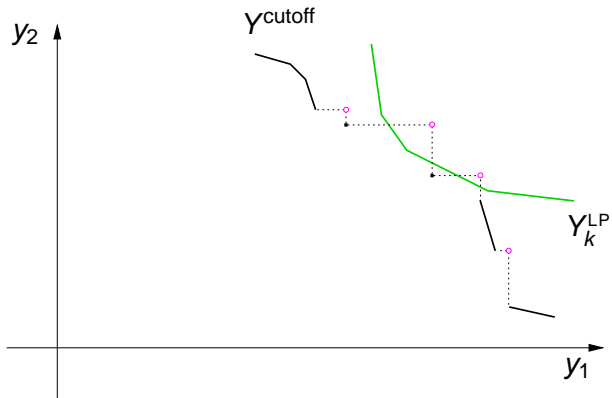
- ▶ Consider the Pareto set Y_k^{LP} of the LP relaxation of the BO-MIP at node k

Then a sufficient condition for fathoming k is that each point \mathbf{y} of Y_k^{LP} is dominated by at least one point of Y^{cutoff} .

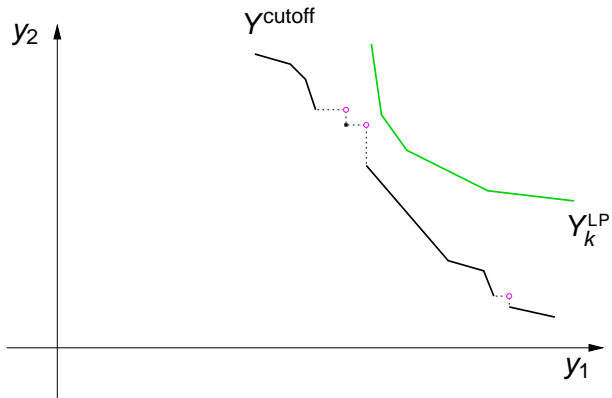
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Nadir points and segments

- ▶ Every \circ in the picture is a **nadir point**.
- ▶ We use nadir points and *segments* to create a sufficient condition for fathoming:

procedure FATHOM ($Y_k^{\text{LP}}, Y^{\text{cutoff}}$)

fathom \leftarrow true

for each nadir point \check{y} **do**

Find line separating $Y_k^{\text{LP}} + \mathbb{R}_+^2$ from $\{\check{y}\} - \mathbb{R}_+^2$

if no such line exists **then**

fathom \leftarrow false

break

for each segment (y', y'') **do**

[...]

if fathom = true **then**

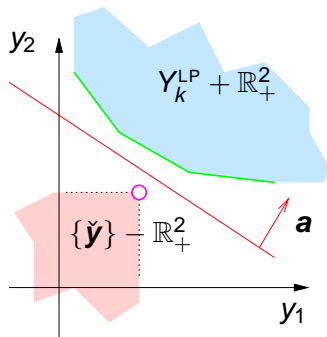
fathom node

Fathoming rule: nadir points \check{y}

The separating line has coefficient vector $\mathbf{a} = (\lambda, 1 - \lambda)$.
Hence we seek $\lambda \in [0, 1]$ such that

$$\lambda \check{y}_1 + (1 - \lambda) \check{y}_2 \leq \lambda y_1 + (1 - \lambda) y_2$$

for every $\mathbf{y} = (y_1, y_2) \in Y_k^{\text{LP}}$



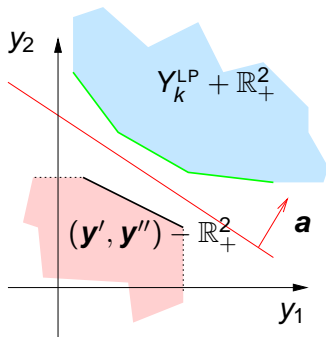
Fathoming rule: segments $(\mathbf{y}', \mathbf{y}'')$

The separating line has coefficient vector $\mathbf{a} = (\lambda, 1 - \lambda)$.
Hence we seek $\lambda \in [0, 1]$ such that

$$\lambda y'_1 + (1 - \lambda)y'_2 \leq \lambda y_1 + (1 - \lambda)y_2$$

$$\lambda y''_1 + (1 - \lambda)y''_2 \leq \lambda y_1 + (1 - \lambda)y_2$$

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Fathoming rule: segments $(\mathbf{y}', \mathbf{y}'')$

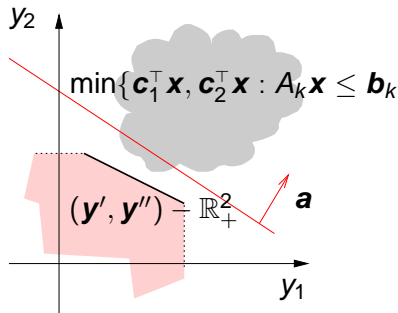
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for every $\mathbf{y} = (y_1, y_2) \in Y_k^{\text{LP}}$

Wait ... we don't have Y_k^{LP}



Fathoming rule: model (nadir points only)

In general: can separate nadir point $\check{\mathbf{y}}$ if

$$\lambda\check{\mathbf{y}}_1 + (1 - \lambda)\check{\mathbf{y}}_2 \leq \lambda\mathbf{c}_1^\top \mathbf{x} + (1 - \lambda)\mathbf{c}_2^\top \mathbf{x} \quad \forall \mathbf{x} \in X_k^{\text{LP}}$$

that is, if we can find a $\lambda \in [0, 1]$ such that

$$\begin{aligned} \lambda\check{\mathbf{y}}_1 + (1 - \lambda)\check{\mathbf{y}}_2 &\leq \min\{\lambda\mathbf{c}_1^\top \mathbf{x} + (1 - \lambda)\mathbf{c}_2^\top \mathbf{x} : \mathbf{x} \in X_k^{\text{LP}}\} \\ &= \min\{\lambda\mathbf{c}_1^\top \mathbf{x} + (1 - \lambda)\mathbf{c}_2^\top \mathbf{x} : A_k \mathbf{x} \leq \mathbf{b}_k\} \\ &= \max\{\mathbf{b}_k^\top \mathbf{w} : A_k^\top \mathbf{w} = \lambda\mathbf{c}_1 + (1 - \lambda)\mathbf{c}_2, \mathbf{w} \leq \mathbf{0}\} \end{aligned}$$

Fathoming rule: model (nadir points only), cont'd

Hence, if we find \mathbf{w} , λ such that

$$\begin{aligned}\lambda\check{\mathbf{y}}_1 + (1 - \lambda)\check{\mathbf{y}}_2 &\leq \mathbf{b}_k^\top \mathbf{w} \\ \mathbf{A}_k^\top \mathbf{w} &= \lambda \mathbf{c}_1 + (1 - \lambda) \mathbf{c}_2 \\ \mathbf{w} &\leq \mathbf{0} \\ \lambda &\in [0, 1]\end{aligned}$$

we can separate $\check{\mathbf{y}}$ from $Y_k^{\text{LP}} + \mathbb{R}_+^2$

Fathoming rule: separation of nadir points

$$\begin{aligned} \min \quad & \lambda \check{y}_1 + (1 - \lambda) \check{y}_2 - \mathbf{b}_k^\top \mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}_k^\top \mathbf{w} = \lambda \mathbf{c}_1 + (1 - \lambda) \mathbf{c}_2 \\ & \mathbf{w} \leq \mathbf{0} \\ & \lambda \in [0, 1] \end{aligned}$$

If optimal solution has positive objective function, we cannot fathom k . Otherwise, check the next nadir point.

- ▶ In theory, must solve an LP for every nadir point \check{y}
- ▶ In practice, we can limit the actual number of LPs solved
- ▶ For segments, we need an extra constraint

Fathoming rule: separation of nadir segment $[y', y'']$

$$\begin{aligned} \min \quad & \lambda \check{y}'_1 + (1 - \lambda) \check{y}'_2 - \mathbf{b}_k^\top \mathbf{w} \\ \text{s.t.} \quad & \lambda \check{y}''_1 + (1 - \lambda) \check{y}''_2 \leq \mathbf{b}_k^\top \mathbf{w} \\ & \mathbf{A}_k^\top \mathbf{w} = \lambda \mathbf{c}_1 + (1 - \lambda) \mathbf{c}_2 \\ & \mathbf{w} \leq \mathbf{0} \\ & \lambda \in [0, 1] \end{aligned}$$

If an optimal solution has nonpositive objective function value, proceed to the next nadir point/segment. If the problem is infeasible or has a positive optimal objective, the node cannot be fathomed.

Fathoming rule: one last simplification

Fathoming LP:

$$\begin{aligned} -\check{y}_2 + \max \quad & \mathbf{b}_k^\top \mathbf{w} + (\check{y}_2 - \check{y}_1)\lambda \\ \text{s.t.} \quad & A_k^\top \mathbf{w} + (\mathbf{c}_2 - \mathbf{c}_1)\lambda = \mathbf{c}_2 \quad (\mathbf{x}) \\ & \lambda \leq 1 \quad (\mu) \\ & \mathbf{w} \leq \mathbf{0}, \lambda \geq 0. \end{aligned}$$

Its dual is somewhat familiar:

$$\begin{aligned} -\check{y}_2 + \min \quad & \mathbf{c}_2^\top \mathbf{x} + \mu \\ \text{s.t.} \quad & A_k \mathbf{x} \leq \mathbf{b}_k \\ & (\mathbf{c}_2 - \mathbf{c}_1)^\top \mathbf{x} + \mu \geq (\check{y}_2 - \check{y}_1) \\ & \mu \geq 0 \end{aligned}$$

It's the original node problem amended with one variable and one constraint (only slightly more involved for segments).

Speeding up fathoming

Large problems \Rightarrow many nadir points available at every node.

- ▶ Restrict to nadir pts. dominated by ideal point of node LP
- ▶ The solution (\mathbf{x}^*, μ^*) to one fathoming LP

$$\begin{aligned} -\check{y}_2 + \min \quad & \mathbf{c}_2^\top \mathbf{x} + \mu \\ \text{s.t.} \quad & \mathbf{A}_k \mathbf{x} \leq \mathbf{b}_k \\ & (\mathbf{c}_2 - \mathbf{c}_1)^\top \mathbf{x} + \mu \geq (\check{y}_2 - \check{y}_1) \\ & \mu \geq 0 \end{aligned}$$

can be used to exclude any other nadir point $\bar{\mathbf{y}}$ such that

$$\begin{aligned} -\bar{y}_2 + \mathbf{c}_2^\top \mathbf{x}^* + \mu^* &\leq 0 \\ (\mathbf{c}_2 - \mathbf{c}_1)^\top \mathbf{x}^* + \mu^* &\geq (\bar{y}_2 - \bar{y}_1) \end{aligned}$$

- ▶ If condition not met for $\bar{\mathbf{y}}$, try a more suitable $\mu \neq \mu^*$.

Used a commercial MILP solver. Added callbacks for

- ▶ **node solving**: used the parametric simplex algorithm (PSA)
- ▶ **branching**: branching variable found during PSA
- ▶ **presolve**: for fathoming rules

Also, disabled cuts and presolving that use objective information (the solver must be oblivious of the two objectives).

A few experiments

var	con	\mathbb{R}	0-1	int	CPU(s)	nodes	infeas	fathom	PSA
20	20	10	5	5	0.62	78.0	22.0	6.0	52.8
20	20	10	10	0	0.54	70.0	21.0	5.4	46.2
20	20	10	0	10	0.56	66.0	21.4	5.6	41.4
40	40	20	10	10	3.32	238.0	67.6	34.6	136.8
40	40	20	20	0	2.38	164.4	38.2	30.0	97.0
40	40	20	0	20	4.00	281.6	61.6	55.0	162.2
60	60	30	15	15	10.86	458.4	108.4	74.6	251.2
60	60	30	30	0	14.38	638.4	171.0	89.8	338.2
60	60	30	0	30	15.90	738.8	212.4	70.4	392.8
80	80	40	20	20	22.34	616.8	156.0	71.2	328.4
80	80	40	40	0	37.58	1026.8	203.6	165.6	538.2
80	80	40	0	40	53.12	1414.8	279.4	219.2	738.4

The alternative

Compare with the ϵ -constraint method:

procedure ϵ -CONSTRAINT ($\mathbf{c}_1, \mathbf{c}_2, A, \mathbf{b}$)

for $\epsilon \in \{\epsilon_1, \epsilon_2, \dots, \epsilon_q\}$ **do**

Solve $\min\{\mathbf{c}_1^\top \mathbf{x} : \mathbf{c}_2^\top \mathbf{x} \leq \epsilon, \mathbf{x} \in X \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})\}$

Save optimal solution \mathbf{x}^* as a Pareto point

Problems:

- ▶ It only works on discrete Pareto sets
- ▶ It is $\approx 20\times$ slower
- ▶ (it requires q single-objective MIPs ...)

Storing a large number of Pareto points (and the corresponding solutions) requires a data structure with

- ▶ fast insertion (with elimination of dominated points)
- ▶ fast check for dominance

We are integrating the branch-and-bound implementation above with a data structure that can be used for fast insertion of new points and that holds the Pareto set at the end of the BB¹.

¹Adelgren, B., Gupte, Efficient storage of Pareto points in bi-objective mixed integer programming, in preparation.

P. Belotti, B. Soylu, M.M. Wiecek, “A branch-and-bound algorithm for biobjective mixed-integer programs”.

http://www.optimization-online.org/DB_HTML/2013/01/3719.html

Thank You

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Decorative wavy lines in various shades of blue are located at the bottom of the slide, creating a sense of motion and depth.