Robust and Multiobjective Optimisation: Opportunities and Challenges

Recent Advances in Multi-Objective Optimization

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Robust Optimisation

Multi-Objective Optimisation

- optimise with several objectives

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$$x \in \mathcal{X}$$

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In this talk:

What can MOO and RO learn from each other?

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2. considering the robust counterpart of a multi-objective problem
3. using a multi-objective perspective on a robust problem

Adding robustness as a new objective function

Idea

$$
\begin{aligned}
\min\ & f(x,\xi) \\
& x \in \mathcal{X}(\xi)
\end{aligned}
$$

Idea

$$\begin{aligned} \min\ & f(x,\xi) \\ & x \in \mathcal{X}(\xi) \end{aligned} \quad \rightarrow \quad \begin{aligned} \min\ & \tilde{f}(x) \\ \max\ & rob(x) \\ & x \in \mathcal{X} \end{aligned}$$

Examples

- Mulvey, Vanderbei, Zenios. *Robust Optimization of Large-Scale Systems*, Oper Res '95.
- Liebchen, Lübbecke, Möhring, Stiller. *The Concept of Recoverable Robustness, Linear Programming Recovery, and Railway Applications*, LNCS 5868, '09.
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$$\begin{aligned} \min\ & f(x,\xi) \\ & F(x,\xi) \le 0 \\ & x \in \mathcal{X} \end{aligned}$$

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$$\begin{aligned} \min\ & f(x,\xi) \\ & F(x,\xi) \le 0 \\ & x \in \mathcal{X} \end{aligned} \quad \rightarrow \quad \begin{aligned} \min\ & \begin{pmatrix} \sup_{\xi\in\mathcal{U}} f(y(\xi),\xi) \\ \sup_{\xi\in\mathcal{U}} d(x,y(\xi)) \end{pmatrix} \\ & F(y(\xi),\xi) \le 0 \ \forall \xi \in \mathcal{U} \\ & x \in \mathcal{X} \\ & y : \mathcal{U} \to \mathcal{X} \end{aligned}$$

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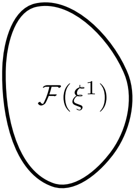

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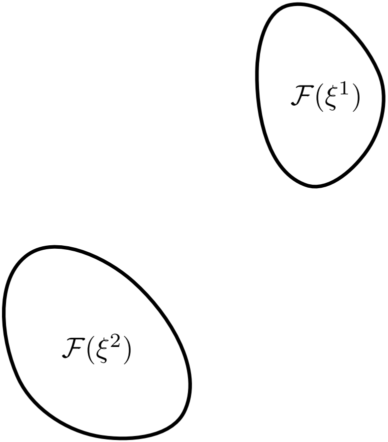

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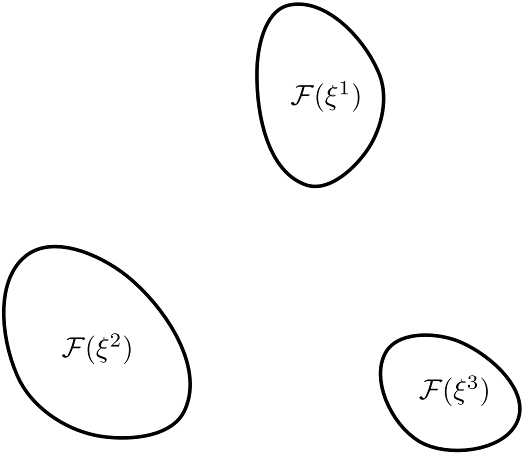

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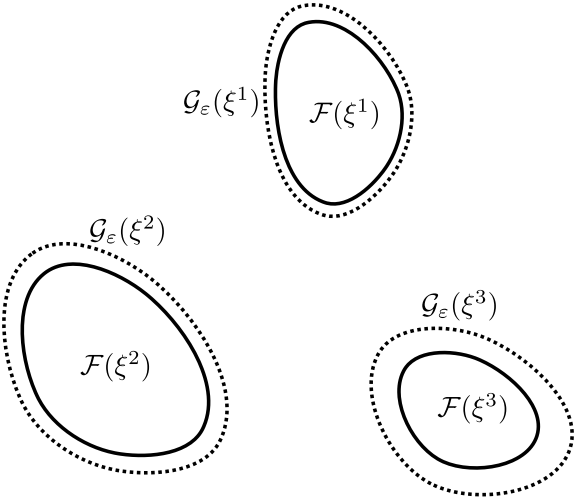

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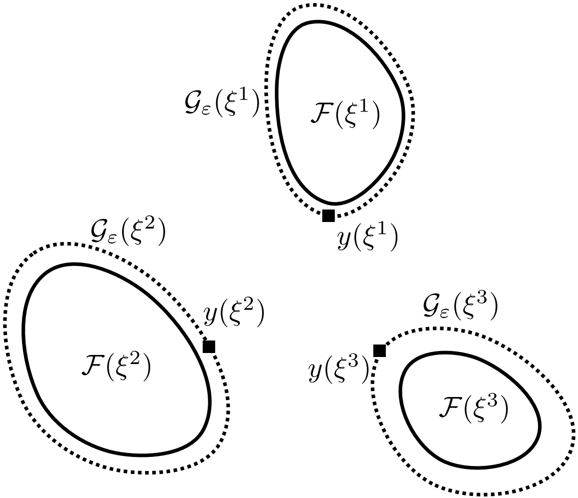

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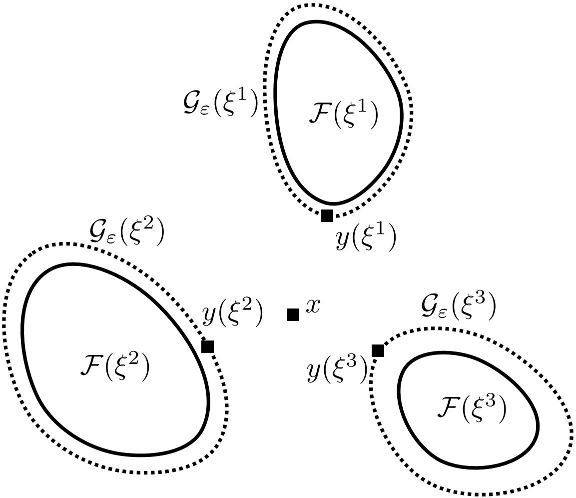

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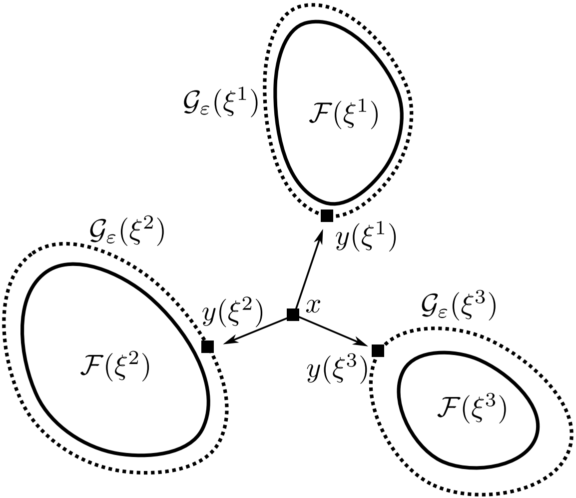

Results on Biobjective Problem

Let

- $\mathcal{U} = conv(\mathcal{U}')$ with $\mathcal{U}' := \{\xi^1, \ldots, \xi^N\}$.
- F consist of m constraints with $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \ldots, m$
- $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ jointly quasiconvex in (y, ξ)
- $d(x, \cdot)$ quasiconvex.
- $\mathcal{X}$ convex

Then Rec($\mathcal{U}$) and Rec($\mathcal{U}'$) have the same set of recoverable-robust solutions.

Approach Summary

- widely used
- most often:

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  - sum of objectives, one arbitrary scaling factor
  - one arbitray budget on (robustness) objective

Approach Summary

- widely used
- most often:
  - sum of objectives, one arbitrary scaling factor
  - one arbitray budget on (robustness) objective
- no effort to find (all/most) Pareto solutions
- multi-objective nature not acknowledged

Robust counterparts of multi-objective problems

Examples

- recently developed
- Ehrgott, Ide, Schöbel. *Minmax robustness for multi-objective optimization problems*, EJOR '14.
- Kuhn, Raith, Schmidt, Schöbel. *Bi-objective robust optimisation*, EJOR '16.
- Deb, Gupta. *Introducing Robustness in Multi-Objective Optimization*, Evol Comp '06.
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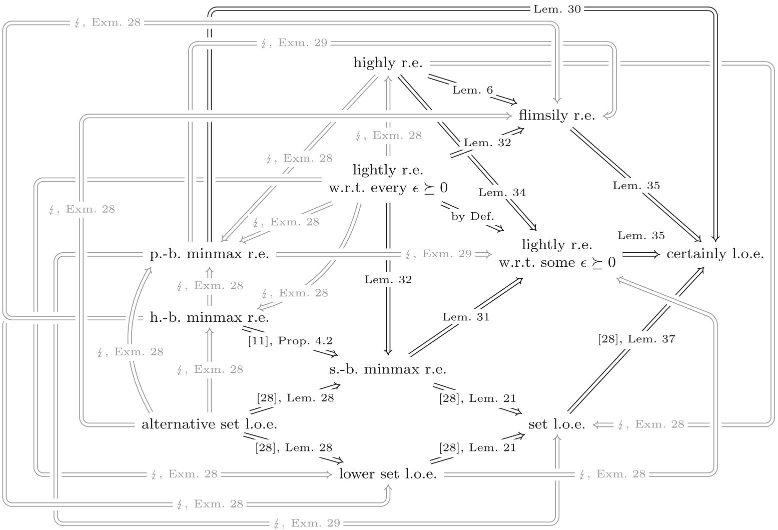
RC of MOP



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Approach Summary

- young and increasingly popular
- see Gabi's talk
- see Corinna's talk

Using a multi-objective perspective on robust problems

Examples

- Aissi, Bazgan, Vanderpooten. *Approximation of min–max and min–max regret versions of some combinatorial optimization problems*, EJOR '07.
  - FPTAS based on FPTAS for multi-objective problems
- Chassein, Goerigk. *A Bicriteria Approach to Robust Optimization*, COR '16.
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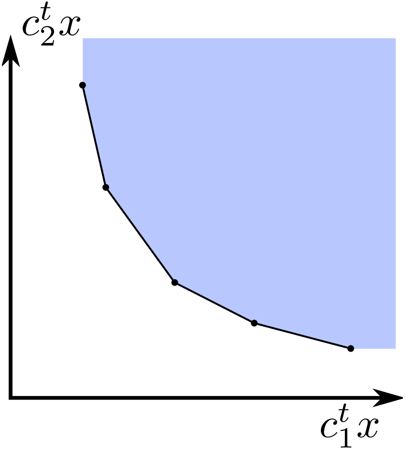

MOO for RO

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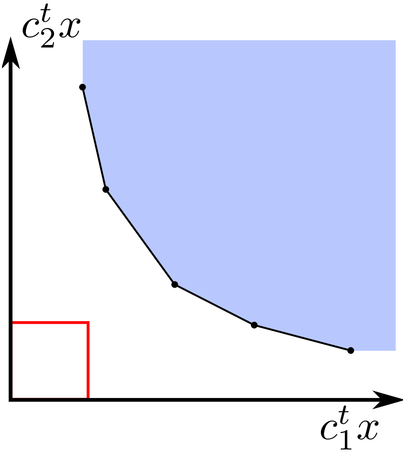

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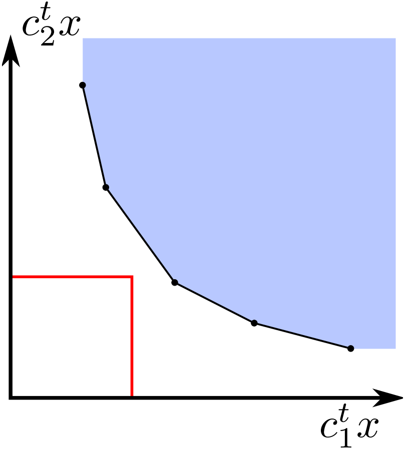

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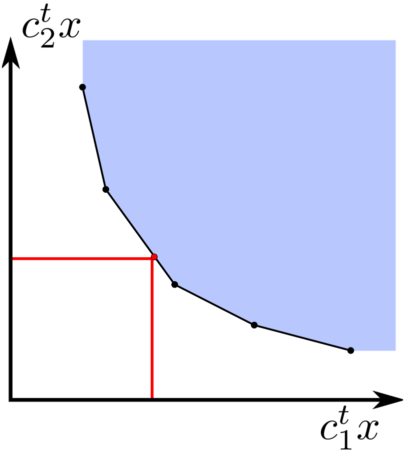
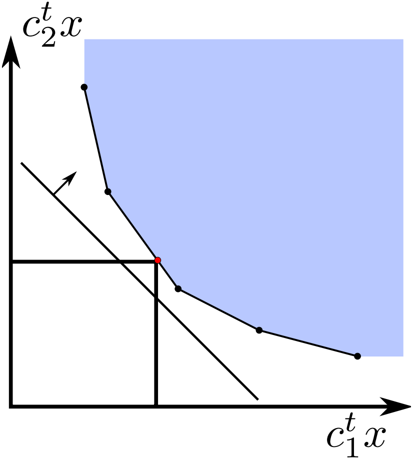


How to use this relationship?

For approximation algorithms:

- average solution
  - $\min c_1^t x + c_2^t x$



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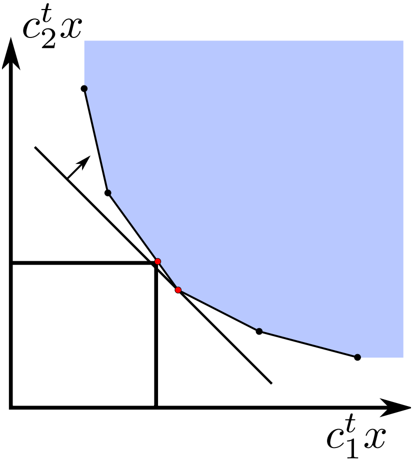

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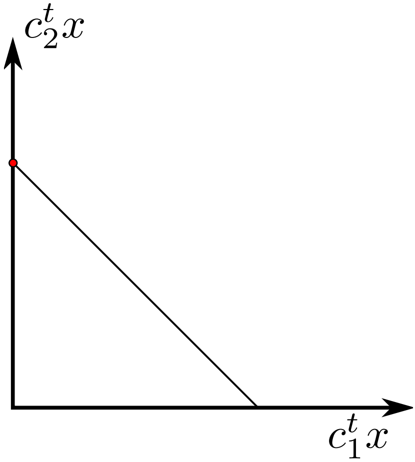

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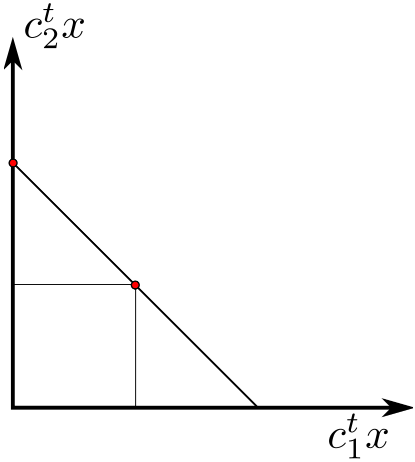

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- quadratic solution
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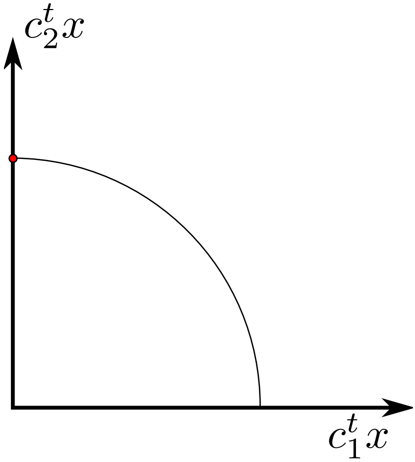

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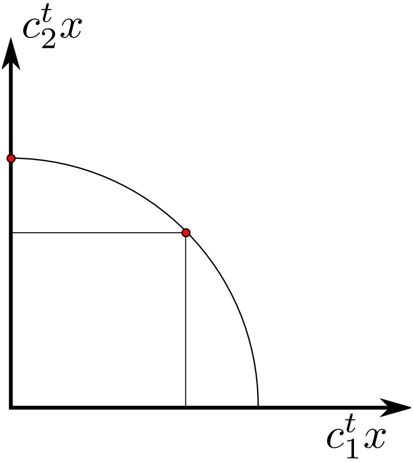
  - $\sqrt{N}$ approx



Computing the AC-WC Curve

Approach to Use Relationship

Consider again our first connection between robust and multi-objective optimisation, using robustness as an additional objective:

$$\min \begin{pmatrix} \hat{c}x \\ \max_{k \in [N]} c^k x \end{pmatrix}$$
$$x \in \mathcal{X}$$

Observations

- As for the WC-solution, we can show:
  - The set of Pareto efficient solution w.r.t. $(c_1, \ldots, c_N, \hat{c})$ contains the Pareto efficient solutions w.r.t. $(z, \hat{c})$

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- Solving weighted sums w.r.t. $(c_1, \ldots, c_N, \hat{c})$ is easy (algorithms of original problem type can be used), but w.r.t. $(z, \hat{c})$ is hard (algorithms for robust problem must be used)

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Idea

Solve "the right" subproblems w.r.t. $(c_1, \ldots, c_N, \hat{c})$ instead of $(z, \hat{c})$!

Master problem

Let $\{y^1, \ldots, y^r\}$ denote the extreme points of all Pareto efficient solutions w.r.t. $(c_1, \ldots, c_N, \hat{c})$ in the objective space.

Master problem

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$$
\begin{aligned}
\text{(M)} \qquad & \min \ \lambda a + (1-\lambda) z \\
& \text{s.t.} \ \hat{y} = \sum_{i=1}^{r} \alpha_i y^i \\
& \sum_{i=1}^{r} \alpha_i = 1 \\
& \hat{y}_{n+1} = a \\
& \hat{y}_i \le z \qquad i = 1, \dots, N \\
& \alpha \ge 0
\end{aligned}
$$

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Problem (M) potentially contains exponentially many variables.

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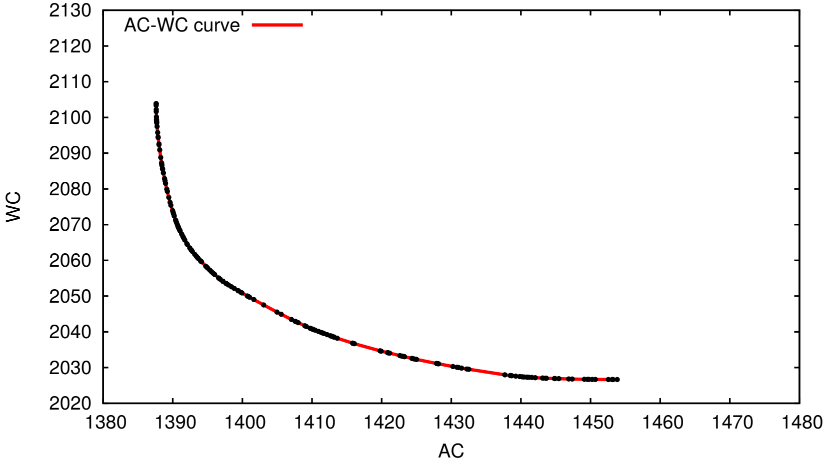
Result

We can compute the AC-WC Pareto front by solving problems of type (M) and problems of the original type.

Computing the AC-WC Curve



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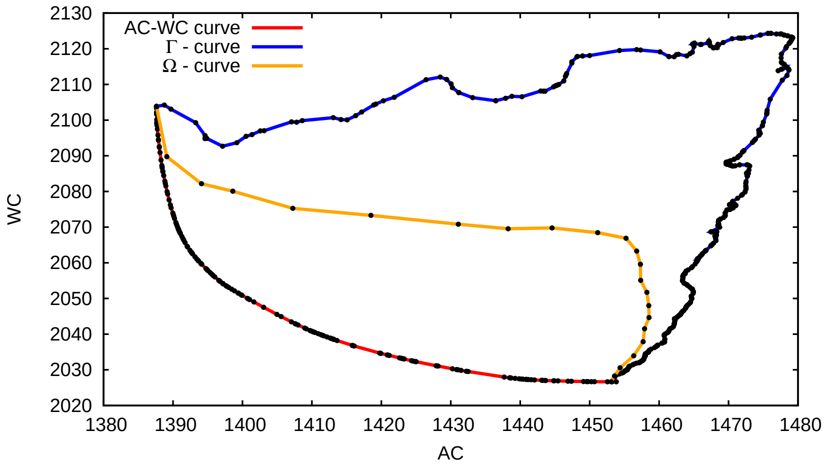

Variable-Sized Uncertainty

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with $\mathcal{U} = \hat{c} + \lambda B$

- We know the shape of uncertainty.
- We do not know the size of uncertainty.
- Find smalles set $\mathcal{S}$ that contains an optimal robust solution for every λ.

Connection to MOO

- Let $B = \prod_{i \in [n]} [-d_i, d_i]$.

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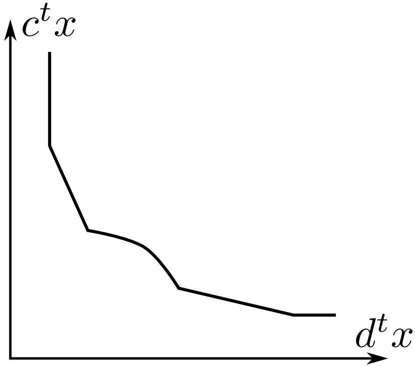

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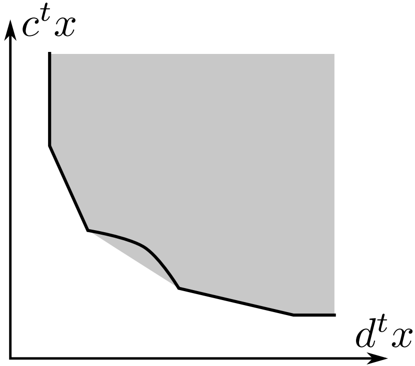

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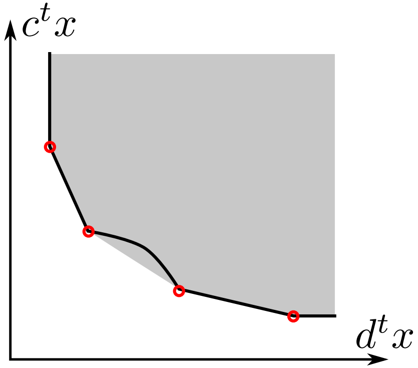

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- Let $B = \prod_{i\in[n]}[-d_i, d_i]$.

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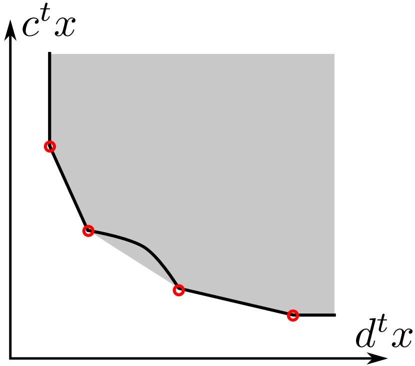
- Find minimal set of efficient extreme points!



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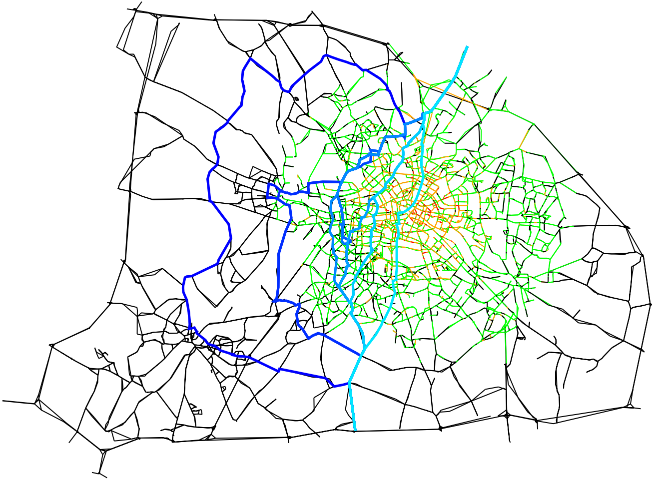
Variable-Sized Uncertainty

Application to Shortest Path

- Compute K efficient points in $O(K)$
- General: $K \in 2^{\Omega(\log^2(n))}$
- Acyclic: $K \in O(n^{\log n})$
- Layered graph: $K \in O(w^{\log(\ell+1)})$
- Series-parallel: $K \in O(m - n)$

Variable-Sized Uncertainty



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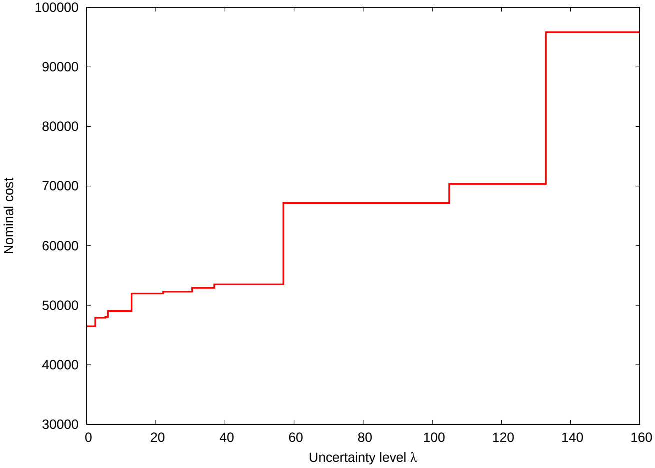

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Overall Summary

Robust and MO Optimisation

- strengthen RO via MOO
- strengthen MOO via RO

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Robust and MO Optimisation

- strengthen RO via MOO
- strengthen MOO via RO
- Plenty of combinations possible!
- Many more exciting combinations to come!