Multiobjective bilevel optimization: A set-valued optimization view point

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Bilevel optimization problem

Conceptual definition:

$$\text{“}\min_{x}\text{”}\ F(x,y) \ \text{ s.t. } \ x \in X,\ y \in S(x)$$

where the map S is defined by

$$S(x) := \arg\min_{y} \{f(x,y) : \ y \in K(x)\}$$

Case where problem is well-posed

(i.e., $S(x) = \{y(x)\}$ for all $x \in X$):

$$(P_i) \qquad \min_{x} F(x, y(x)) \ \text{ s.t. } \ x \in X$$

Outrata et al. (1998), Dempe (2002), Falk & Liu (1995), Kolstad & Ladson (1990), Savard & Gauvin (1994), Vicente et al. (1994), Mersha (2011), etc.

Forcing uniqueness in the lower-level problem

$$\min_{y} f(x,y) + \alpha\pi(x,y) \quad \text{s.t.} \quad g(x,y) \le 0 \qquad (\alpha > 0)$$

Example: $\pi(x,y) := \|y\|^2$ (Tikhonov regularization)

See Dempe & Schmidt (1996), Dempe & Bard (2001), Morgan & Patrone (2006), Bergounioux & Haddou (2008), Molodtsov (1976), etc.

Purpose of the talk

Set-valued optimization model:

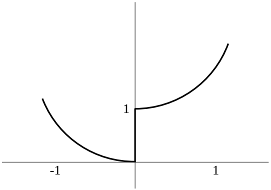
$$(P_s) \qquad \min_x \mathcal{F}(x) := F(x, S(x)) \text{ s.t. } x \in X$$

Example by Lucchetti et al. (1987):

$$F(x,y) := x^2 + y^2,\ X := [-1,1]$$

$$S(x) := \arg\min_y \{-xy :\ y \in [0,1]\}$$

$$S(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ [0,1] & \text{if } x = 0 \end{cases}$$



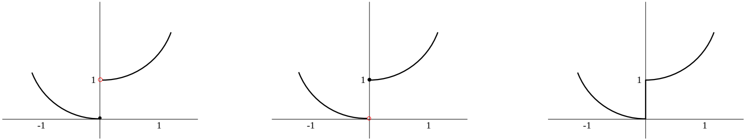
(c) graph of $\mathcal{F}$

$(\bar{x}, \bar{z}) \in \operatorname{gph} \mathcal{F} := \{(x,z) \in \mathbb{R}^n \times \mathbb{R} :\ z \in \mathcal{F}(x)\}$ is a local Pareto optimal solution of (P_s) if there exists a neighborhood U of $\bar{x}$ such that

$$(\mathcal{F}(X \cap U) - \bar{z}) \cap]-\infty, 0[= \emptyset.$$

How can the set-valued model be useful?

- (P) $\quad \min\limits_{x,y} F(x,y)$ s.t. $x \in X,\ y \in S(x)$
- (P_o) $\quad \min\limits_{x \in X} \varphi_o(x) := \min\limits_{y} \{F(x,y) |\ y \in S(x)\}$
- (P_p) $\quad \min\limits_{x \in X} \varphi_p(x) := \max\limits_{y} \{F(x,y) |\ y \in S(x)\}$



(a) graph of φ_o (b) graph of φ_p (c) graph of $\mathcal{F}$

$$(P_p) \overset{(**)}{\Longleftrightarrow} (P_i) \overset{(**)}{\Longleftrightarrow} (P_o) \Longleftrightarrow (P_s)$$

$$\Updownarrow (*)$$

$$(P)$$

Another perspective

Bilevel optimization as multiobjective optimization

- Marcotte and Savard (1991)
- Fülöp (1993)
- Fliege and Vicente (2006)
- Ruuska, Miettinen and Wiecek (2012), etc.

Related solution approaches

- Glackin, Ecker and Kupferschmid (2009)
- Alves, Dempe and Júdice (2010)
- Pieume, Fotso and Siarry (2009), etc.

Other set-valued related bilevel models

- Bonnel (2006), Bonnel and Morgan (2012)
- Eichfelder (2008), Ye (2011)
- Dempe and Gadhi (2010), Dempe and Pilecka (2015), etc.

Summary of results for scalar objectives

Theorem: The following properties are satisfied:

(a) $(\text{gph}\,\varphi_o \cup \text{gph}\,\varphi_p) \cap (X \times \mathbb{R}) \subseteq \text{gph}\,\mathcal{F}$.

(b) Let $F(x, y) := a(x)^\top y + b(x)$ with $a : \mathbb{R}^n \to \mathbb{R}^m$ and $b : \mathbb{R}^n \to \mathbb{R}$, and assume that S is convex-valued on X^m. Then, for all $x \in X$,

$$\mathcal{F}(x, S(x)) \begin{cases} = \varphi_o(x) = \varphi_p(x) & \text{if } x \in X^u \\ \supseteq [\varphi_o(x), \varphi_p(x)] & \text{if } x \in X^m \end{cases}$$

Theorem: $\bar{x}$ is an optimal solution of problem (P_o) if and only if there exists a vector $\bar{z} \in \mathcal{F}(\bar{x})$ such that $(\bar{x}, \bar{z})$ is a Pareto optimal solution of problem (P_s).

Theorem: Let $(\bar{x}, \bar{z}) \in \text{gph}\mathcal{F}$ be a local Pareto optimal solution of (P_s) and assume that S is closed, locally bounded around $\bar{x}$ and

$$D^* S(\bar{x}|y)(0) \cap (-N_X(\bar{x})) = \{0\} \;\; \forall y \in S(\bar{x}) \;\; \text{s.t. } \bar{z} = F(\bar{x}, y).$$

Then, there exists $\bar{y} \in S(\bar{x})$ with $\bar{z} = F(\bar{x}, \bar{y})$ such that we have

$$-\nabla_x F(\bar{x}, \bar{y}) \in D^* S(\bar{x}|\bar{y})(\nabla_y F(\bar{x}, \bar{y})) + N_X(\bar{x}).$$

Results for vector-valued objectives

$F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^{\ell_1}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^{\ell_2}$

$$S_{wef}(x) = \bigcup_{y \in Y} S_s(x, y) := \arg\min_z \left\{ \langle y, f(x, z) \rangle : \ z \in K(x) \right\}$$

Proposition: Assume that S_s is closed and let $\bar{z} \in S_{wef}(\bar{x})$. Further suppose

$$\left[(0, -u) \in D^* S_s((\bar{x}, \bar{y})|\bar{z})(0), \ u \in U(\bar{y}) \right] \Longrightarrow u = 0 \qquad (1)$$

hold for all $\bar{y} \in Y$ such that $\bar{z} \in S_s(\bar{x}, \bar{y})$. Then for all $z^* \in \mathbb{R}^m$, we have

$$D^* S_{wef}(\bar{x}|\bar{z})(z^*) \subseteq \Big\{ x^* : \ (x^*, -\bar{u}) \in D^* S_s((\bar{x}, \bar{y})|\bar{z})(z^*), \\ \bar{y} \in Y, \ \bar{u} \in U(\bar{y}), \ \bar{z} \in S_s(\bar{x}, \bar{y}) \Big\}.$$

Theorem: Let $(\bar{x}, z^o) \in \mathrm{gph}\mathcal{F}$ be a local Pareto solution of (P_s) and let S_s be closed and locally bounded near $\bar{x}$. Further let QC (1)

$$\left[(x^*, -u) \in D^* S_s((\bar{x}, y)|z)(0), \ \ x^* \in -N_X(\bar{x}) \right] \Longrightarrow x^* = 0.$$

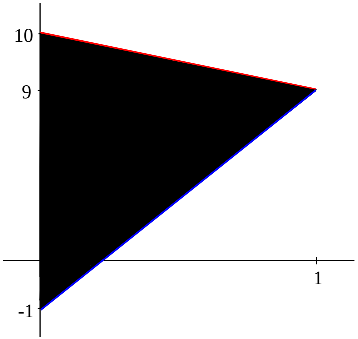
Then, there exist $z^* \in \mathbb{R}^{l_2}$ with $\|z^*\| = 1$ and a vector $(\bar{y}, \bar{u}, \bar{z})$ such that

$$-\left(\nabla_x F(\bar{x}, \bar{z})^\top z^*, \bar{u}\right) \in D^* S_s((\bar{x}, \bar{y})|\bar{z})(\nabla_z F(\bar{x}, \bar{z})^\top z^*) + N_X(\bar{x}) \times \{0_{l_2}\}.$$

Illustrative examples

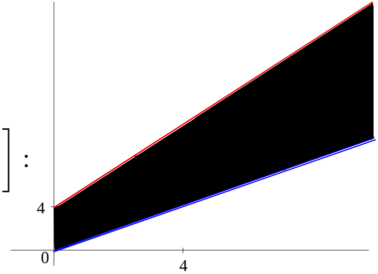
Scalar objectives example

$$
\begin{aligned}
F(x,y) &:= -x + 10y_1 - y_2 \\
X &:= [0,\infty[\\
S(x) &:= \arg\min_y \{-y_1 - y_2 : \\
&\qquad\qquad y_1,\ y_2 \geq 0 \\
&\qquad\qquad x - y_1 \leq 1 \\
&\qquad\qquad x + y_2 \leq 1 \\
&\qquad\qquad y_1 + y_2 \leq 1\}
\end{aligned}
$$



Semivectorial example

$$
\begin{aligned}
F(x,z) &:= x + z_2 \\
X &:= [0,\infty[\\
S(x) &:= \arg\min_z \left\{ \begin{bmatrix} 2z_1 + 2z_2 \\ -z_1 + z_2 \end{bmatrix} : \right. \\
&\qquad \left. \begin{array}{r} z_1,\ z_2 \geq 0 \\ -x + z_1 + z_2 \geq 4 \end{array} \right\}
\end{aligned}
$$



Ongoing & future research topics

- Solution algorithms based on set-valued optimization
  - Generalization of the implicit function model techniques
  - Further analysis of the map $\mathcal{F}(x) := F(x, S(x))$
- Implementation of the methods on the pessimistic model

Alain B. Zemkoho (2016). Solving ill-posed bilevel programs. Set-Valued and Variational Analysis, *in press.*