Ress Eff GRK 1703

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

On Decision Uncertainty in Multiobjective Linear Programming

Gabriele Eichfelder[1], Corinna Krüger[2,*)], Anita Schöbel[2], Margaret Wiecek[3]

[1] University of Ilmenau, Germany
[2] University of Göttingen, Germany
[3] Clemson University, USA
*) Member of DFG-RTG 1703

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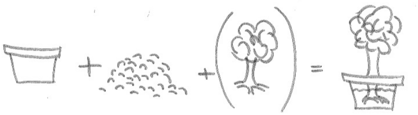
Decision Uncertainty: How the story started...

Example: Optimizing soil for potted plants in horticultural industries



Problem

What is a “good” mixing ratio for combining different kinds growing media for a plant?

Decision Variables mixing ratio per plant species

Objectives minimizing negative environmental impact, maximizing profit

- Putting a solution into practice causes problems:
  - technical limitations and workers will rather work fast that very exact
- realizations of decision variables are uncertain
- objective value of a realized decision variable is possibly not efficient

Decision Uncertainty in Multiobjective Optimization

Let

$$\begin{array}{lll} (\mathrm{P}) & \min & C\,x \\ & \text{s. t.} & A \cdot x \geq b \\ & & x \in \mathbb{R}^N \end{array}$$

be a linear multiobjective optmization problem, where $C \in \mathbb{R}^{p \times n}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

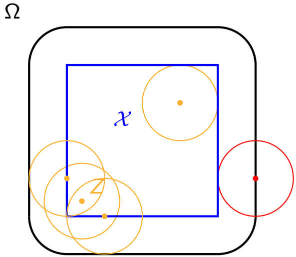
To account for decision uncertainty in (P), we consider the corresponding uncertain problem:

$$\left\{ \begin{array}{lll} (\mathrm{P}(z)) & \min & C\,(x+z) \\ & \text{s. t.} & A \cdot (x+z) \geq b \\ & & x \in \mathbb{R}^N \end{array} \right\}_{z \in Z} .$$

Decision Robust Feasibility

We consider

- a feasible set $\Omega \subset \mathbb{R}^n$
- a compact set $Z \subset \mathbb{R}^n$ with $0 \in Z$
- definition dates back to [Das97], [LP09].



Definition

A solution $x \in \Omega$ is called *decision robust feasible* with respect to Z if

$$x + z \in \Omega \qquad \forall z \in Z.$$

We denote the *set of decision robust feasible solutions* by

$$\mathcal{X} := \bigcap_{z \in Z} \{x \in \mathbb{R}^n \mid x + z \in \Omega\} \subseteq \Omega.$$

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Decision Robust Efficiency

We consider

- a feasible set $\Omega \subset \mathbb{R}^n$
- a compact set $Z \subset \mathbb{R}^n$ with $0 \in Z$
- the decision robust feasible set $\mathcal{X} \subset \Omega$
- a continuous objective function $f : \Omega \to \mathbb{R}^p$
- the problem

$$(\text{RC}) \qquad \min_{x \in \mathcal{X}} \; f_Z(x).$$

Definition

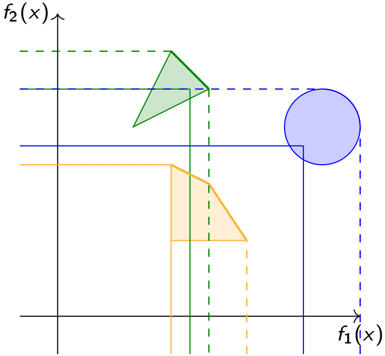
An element $x^* \in X$ is called *decision robust efficient* with respect to Z, if x^* is decision robust feasible and if there is no $x \in \mathcal{X} \setminus \{x^*\}$ with the property

$$f_Z(x) \subseteq f_Z(x^*) - \mathbb{R}^p_+ \setminus \{0\},$$

where $f_Z(x)$ is defined as the set $f_Z(x) = \{f(x+z) \colon z \in Z\}$.

Decision Robustness in Multi-Objective Optimization

Bi-Objective



Decision Uncertainty in Linear Objective Functions

Theorem [EKS15]

Let $f : \Omega \to \mathbb{R}^p, x \mapsto Cx$ be linear. For each decision robust feasible solution $x \in \mathcal{X}$, the following two are equivalent:

$$
\begin{array}{llll}
& x \text{ is efficient for} & (\mathrm{P}|_{\mathcal{X}}) & \min\limits_{x \in \mathcal{X}} \quad Cx \\
\Leftrightarrow & x \text{ is decision robust efficient for} & (\mathrm{RC}) & \min\limits_{x \in \mathcal{X}} f_Z(x).
\end{array}
$$

- Also holds for closed convex pointed cones $K \subseteq \mathbb{R}^p$ instead of $\mathbb{R}^p_+$.

The Robust Counterpart for Decision Uncertainty in MOLP

For the uncertain problem

$$\left\{ \begin{array}{lll} (\mathrm{P}(z)) & \min & C\ (x+z) \\ & \text{s. t.} & A \cdot (x+z) \geq b \\ & & x \in \mathbb{R}^N \end{array} \right\}_{z \in Z} .$$

We hence obtain the *Robust Counterpart*

$$\begin{array}{llll} (\mathrm{RC}) & \min & Cx & \\ & \text{s. t.} & A(x+z) \geq b & \forall z \in Z \\ & & x \in \mathbb{R}^n & \end{array}$$

We rearrange the side conditions

$$\begin{array}{lrl} & A(x+z) \geq b & \forall z \in Z \\ \Leftrightarrow & Ax \geq b - Az & \forall z \in Z \end{array}$$

Define a different uncertainty set

$$\mathcal{U} := \{\xi = b - Az \mid z \in Z\} \subseteq \mathbb{R}^m$$

We finally obtain the *Robust Counterpart* as

$$\begin{array}{llll} (\mathrm{RC}) & \min & Cx & \\ & \text{s. t.} & Ax \geq \xi & \forall \xi \in \mathcal{U} \\ & & x \in \mathbb{R}^n & \end{array}$$

The Robust Counterpart for Decision Uncertainty in MOLP

$$\begin{array}{llll} \text{(RC)} & \min & Cx & \\ & \text{s. t.} & Ax \geq \xi & \forall \xi \in \mathcal{U} \\ & & x \in \mathbb{R}^n & \end{array}$$

- In addition to assuming Z compact, we also require Z to be a convex polyhedral set.
- Hence, the uncertainty set $\mathcal{U} := \{\xi = b - Az \mid z \in Z\} \subseteq \mathbb{R}^m$ is a compact convex polyhedral set as well.
- Then, there exist $D \in \mathbb{R}^{s \times m}$ and $d \in \mathbb{R}^s$ such that

$$\mathcal{U} = \{\xi \in \mathbb{R}^m \mid D\xi \leq d\}.$$

- In the following, we study MOLP with deterministic objective functions and polyhedral uncertainty in the right hand side of the constraints.
- The side conditions of (RC) fit into the robust optimization framework of [BTGN09].

Contents

The Robustness Gap

- Decision makers often want to know how much they "sacrifice" for gaining a robust efficient solution.
- How to measure the different performance of an efficient and a robust efficient solution?

In single-objective optimization, we refer to the well-known concept of [BTGN09].

Definition

Let $g\colon \Omega \to \mathbb{R}$, let an uncertain linear optimization problem (P^{s-o}) and its robust counterpart be given as

$$(P^{s-o}) \quad \{\min\{g(x) \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}} \quad \text{and} \quad (RC^{s-o}) \quad \min\{g(x) \mid Ax \geq \xi,\ \forall \xi \in \mathcal{U}\}.$$

The *single-objective robustness gap* is defined as

$$RGSO := \min_{x\colon Ax \geq \xi \forall \xi \in \mathcal{U}} g(x) - \sup_{\xi \in \mathcal{U}} \min_{x\colon Ax \geq \xi} g(x).$$

Example: Robustness Gap

Let

$$(P^{s-o}) \qquad \min \quad \frac{1}{\sqrt{5}}(x_1 + 2x_2)$$

$$\text{s. t.} \quad x_1 \geq \xi_1, \quad x_2 \geq \xi_2, \quad x_1 + x_2 \geq \xi_3 \qquad \forall \xi \in \mathcal{U}$$

where $\mathcal{U} = \left\{ \begin{pmatrix} 10 \\ 0 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 15 \end{pmatrix} \right\}$.

The corresponding robust counterpart is

$$(RC^{s-o}) \qquad \min \quad \frac{1}{\sqrt{5}}(x_1 + 2x_2)$$

$$\text{s. t.} \quad x_1 \geq 10, \quad x_2 \geq 10, \quad x_1 + x_2 \geq 15$$

The robustness gap is

$$RGSO = \min_{x:\ Ax \geq \xi \forall \xi \in \mathcal{U}} g(x) - \sup_{\xi \in \mathcal{U}} \min_{x:\ Ax \geq \xi} g(x) = \frac{30}{\sqrt{5}} - \frac{25}{\sqrt{5}} = \sqrt{5}.$$

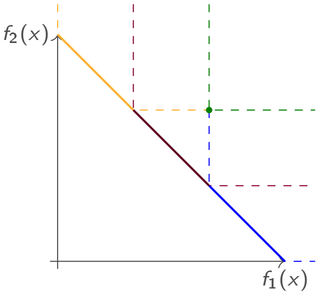
Example: Robustness Gap

Consider a multiobjective problem with uncertain right hand-sides:

$$
\begin{array}{llrl}
(P) & \min & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \\
& \text{s. t.} & x_1 \geq \xi_1 & \\
& & x_2 \geq \xi_2 & \forall \xi \in \mathcal{U} \\
& & x_1 + x_2 \geq \xi_3 &
\end{array}
$$

where $\mathcal{U} = \left\{ \begin{pmatrix} 10 \\ 0 \\ 15 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 15 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 15 \end{pmatrix} \right\}$.

How can we define a robustness gap in multiple dimensions?



Defining a Robustness Gap for MOLP with Decision Uncertainty

Definition

Let $\{(\mathrm{P}(\xi)) \quad \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. We denote the set of decision robust efficient solutions to (RC) by $\mathcal{Y}_P^{RC}$ and for each $\xi \in \mathcal{U}$ we denote the set of efficient solutions to $\mathrm{P}(\xi)$ by $\mathcal{Y}_P(\xi)$.
We define the *Robustness Gap* for $\{(\mathrm{P}(\xi))\}_{\xi \in \mathcal{U}}$ as

$$\vartheta := \min_{\xi \in \mathcal{U}} \left(\min_{y \in \mathcal{Y}_P(\xi)} \min_{z \in \mathcal{Y}_P^{RC}} \|y - z\|_2 \right).$$

- Computing the robustness gap ϑ requires optimizing over the Pareto-sets of the problems $(\mathrm{P}(\xi))$.
- We want to approximate the robustness gap ϑ.
- lower and upper bounds on ϑ?
- Obviously, $\vartheta \geq 0$

A lower bound on the robustness gap ϑ

- We make use of the single-objective definition of the robustness gap by [BTGN09].
- By weighted-sum scalarization, we transform our multiobjective problem $\{(\mathrm{P}(\xi))\}$ into a single-objective problem.

Definition

Let $\{(\mathrm{P}(\xi)) \quad \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. For each $\xi \in \mathcal{U}$ and for each $\lambda \in \mathbb{R}^p_+$, we define

$$\Delta(\xi, \lambda) := \min_{z \in \mathcal{Y}_P^{RC}} \lambda^T z - \min_{y \in \mathcal{Y}_P(\xi)} \lambda^T y.$$

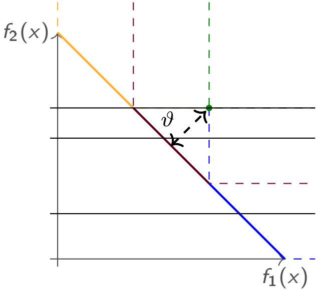
Furthermore, the *lower Robustness Gap* is defined as

$$\Delta := \min_{\xi \in \mathcal{U}} \min_{\substack{\lambda \in \mathbb{R}^p_+, \\ \|\lambda\| = 1}} \Delta(\xi, \lambda).$$

A lower bound on the robustness gap ϑ

In our example, we have

$$\Delta = 0 \quad < \quad \sqrt{2} \cdot 5 = \vartheta.$$



Theorem

Let $\{(\mathrm{P}(\xi)) \quad \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty. The lower robustness gap Δ is indeed a lower bound for the robustness gap ϑ, i.e.,

$$\begin{aligned} \Delta = \quad & \min_{\xi \in \mathcal{U}} \min_{\substack{\lambda \in \mathbb{R}^p_+, \\ \|\lambda\|=1}} \left(\min_{z \in \mathcal{Y}_P^{RC}} \lambda^T z - \min_{y \in \mathcal{Y}_P(\xi)} \lambda^T y \right) \\ \leq \quad \vartheta = \quad & \min_{\xi \in \mathcal{U}} \left(\min_{y \in \mathcal{Y}_P(\xi)} \min_{z \in \mathcal{Y}_P^{RC}} \|y - z\|_2 \right). \end{aligned}$$

The Relationship between the RG ϑ and its lower bound Δ

- In which cases is lower bound Δ a tight approximation of the robustness gap ϑ?

Let $\{(\mathrm{P}(\xi)) \quad \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty.

Theorem

Let there exist $\bar{z} \in \mathcal{Y}_P^{RC}$, $\bar{\xi} \in \mathcal{U}$ and $\bar{y} \in \mathcal{Y}_P(\bar{\xi})$ that satisfy

$$\vartheta = \|\bar{z} - \bar{y}\|_2$$

such that there exists a $p-1$-dimensional face $\mathcal{F}$ of the polyhedral set $\mathcal{Y}(\xi)$ with $\bar{y} \in \mathrm{relInt}(\mathcal{F})$.
Then Δ a tight approximation of the robustness gap ϑ, i.e.,

$$\Delta = \vartheta.$$

The Relationship between the RG ϑ and its lower bound Δ

Let $\{(\mathrm{P}(\xi)) \quad \min\{Cx \mid Ax \geq \xi\}\}_{\xi \in \mathcal{U}}$ be a multiobjective linear problem with decision uncertainty.

Theorem

Let there exist $\bar{z} \in \mathcal{Y}_P^{RC}$, $\bar{\xi} \in \mathcal{U}$ and $\bar{y} \in \mathcal{Y}_P(\bar{\xi})$ that satisfy

$$\vartheta = \|\bar{z} - \bar{y}\|_2$$

such that there exists a $p-1$-dimensional face $\mathcal{F}$ of the polyhedral set $\mathcal{Y}^{RC}$ with $\bar{z} \in \mathrm{relInt}(\mathcal{F})$.
Then Δ a tight approximation of the robustness gap ϑ, i.e.,

$$\Delta = \vartheta.$$

- Furthermore, $\Delta = \vartheta$ whenever the robustness gap ϑ is 0.
- How to determine the lower bound Δ?

Determining the lower bound Δ by Quadratic Programming

Using the assumption $\mathcal{U} = \{\xi \in \mathbb{R}^m \mid D\xi \leq d\}$, we obtain by the dual formultions of (P(ξ)) and (RC):

Define

$$\bar{b} = \left\{ \max_{\xi \in \mathcal{U}} \xi_i \right\}_{i=1}^{m} \in \mathbb{R}^m.$$

$$\begin{aligned} \Delta \;=\; \min \quad & \bar{b}^T u - \xi^T v \\ \text{s. t.} \quad & v^T A = \lambda^T C \\ & D\xi \leq d \\ & u^T A = \lambda^T C \\ & Ax \geq \bar{b} \\ & \lambda^T Cx = \bar{b}^T u \\ & \|\lambda\|_2 = 1 \\ & v, u, \lambda \geq 0 \end{aligned}$$

Thank you for your attention!

References

- A. Ben-Tal, L. El Ghaoui, and A. Nemirovski, *Robust optimization*, Princeton University Press, Princeton and Oxford, 2009.
- Indraneel Das, *Nonlinear multicriteria optimization and robust optimality*, Ph.D. thesis, Rice University, 1997.
- M. Ehrgott, J. Ide, and A. Schöbel, *Minmax robustness for multi-objective optimization problems*, European Journal of Operational Research **239** (2014), 17–31.
- G. Eichfelder, C. Krüger, and A. Schöbel, *Multi-objective regularization robustness*, Tech. Report 2015-13, Preprint-Reihe, Institut für Numerische und Angewandte Mathematik, Georg-August Universität Göttingen, 2015.
- A.S. Lewis and C.H.J. Pang, *Lipschitz behavior of the robust regularization*, SIAM Journal on Control and Optimization **48** (2009), no. 5, 3080–3105.