Multi-objective decision uncertainty and set optimization with the set approach

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June 24, 2016

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Section 1
Decision Uncertainty

Basic idea

Given: function $f : \mathbb{R}^n \to \mathbb{R}$, a feasible set $\Omega \subset \mathbb{R}^n$, and the optimization problem

$$\bar{x} \in \text{argmin}_{x \in \Omega} f(x)$$

but ...

Basic idea

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(e.g. the optimal temperature/height/volume)
$\Rightarrow$ Uncertainty in the realization

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$\Rightarrow$ Uncertainty in the realization

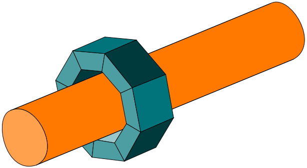
... ... we choose a robust approach (in the sense of Ben-Tal and Nemirovski): "min-max"

Example: Design of a magnetic system

The Lorentz force velocimetry (LFV) is an electromagnetic non-contact flow measurement technique for electrically conducting fluids.
The task is to find an optimal magnet design which

- results in a strong Lorentz force (f_1)
- and has a minimal weight (f_2)

i.e. we have **two** objective functions.



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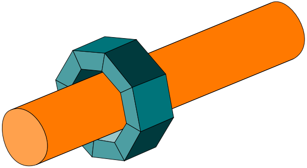
Example: Design of a magnetic system



- variables: direction of the magnetization (angles Φ_i and Θ_i) and the magnetization M_i of each magnet i
- in practice an (optimally) chosen magnetic direction $\overline{\Phi}_i$, $\overline{\Theta}_i$ cannot be realized in the desired accuracy
- only magnets can be ordered with a guarantee of $\Phi_i \in [\overline{\Phi}_i - \varepsilon, \overline{\Phi}_i + \varepsilon]$ and $\Theta_i \in [\overline{\Theta}_i - \varepsilon, \overline{\Theta}_i + \varepsilon]$ for some $\varepsilon > 0$.

Notation and Assumptions

Given: compact set Z of all possible deviations (with $0 \in Z$)
instead of x we have to consider $\{x + z \mid z \in Z\}$

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Given: $f : \mathbb{R}^n \to \mathbb{R}$ continuous,
and a feasible set $\Omega \subseteq \mathbb{R}^n$

Consider for each $x \in \Omega$ the set

$$f_Z(x) := \{f(x + z) \in \mathbb{R} \mid z \in Z\}$$

instead of $f(x)$.

Feasibility

Given: compact set Z of all possible deviations (with $0 \in Z$) and feasible set $\Omega \subseteq \mathbb{R}^n$

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Feasibility

Given: compact set Z of all possible deviations (with $0 \in Z$) and feasible set $\Omega \subseteq \mathbb{R}^n$

Definition (Feasibility, Lewis '02)

A point $x \in \Omega$ is called *decision robust feasible* with respect to Z if

$$\{x + z \mid z \in Z\} \subseteq \Omega,$$

i.e., if all realizations of x are feasible.

Feasible set of the robust problem:

$$X := \{x \in \Omega \mid x + z \in \Omega \text{ for all } \; z \in Z\}$$

(Scalar-valued) Decision Uncertainty

Under the name robust regularization Lewis suggested for a scalar-valued objective function to solve

$$\min_{x \in X} \sup_{z \in Z} f(x+z)$$

i.e. for the map

$$x \mapsto \bar{f}(x) := \sup_{z \in Z} f(x+z)$$

to solve the problem

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But what to do in case of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m > 1$?

Section 2
Multi-Objective Decision Robust Efficiency

Efficient solutions in multi-objective optimization

Consider

$$\min_{x \in \Omega} f(x) \quad \text{with} \quad f\colon \mathbb{R}^n \to \mathbb{R}^m$$

and a pointed convex cone $K \subset \mathbb{R}^m$ (with $\text{int}(K) \neq \emptyset$).
For instance $K = \mathbb{R}^m_+$.

Definition

A solution $\bar{x} \in \Omega$ is called *efficient* if there is no $x \in \Omega$ with $f(x) \in \{f(\bar{x})\} - K \setminus \{0\}$.

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A solution $\bar{x} \in \Omega$ is called *weakly efficient* if there is no $x \in \Omega$ with $f(x) \in \{f(\bar{x})\} - \operatorname{int}(K)$.

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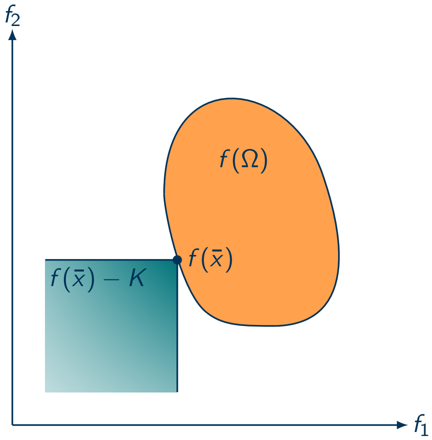
$$\bar{x} \text{ is efficient} \iff (\{f(\bar{x})\} - K) \cap f(\Omega) = \{f(\bar{x})\}$$

$$\bar{x} \text{ is weakly efficient} \iff (\{f(\bar{x})\} - \operatorname{int}(K)) \cap f(\Omega) = \emptyset$$

Efficient solutions



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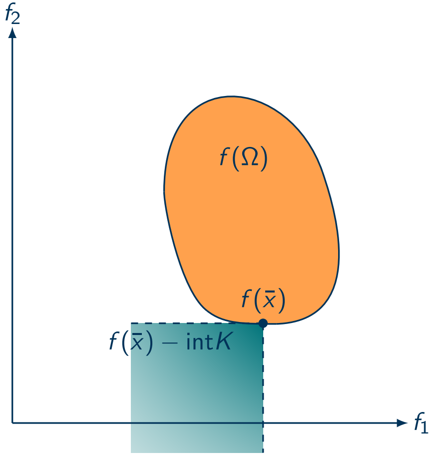


$\bar{x}$ is weakly efficient $\iff (\{f(\bar{x})\} - \mathrm{int}(K)) \cap f(\Omega) = \emptyset$

Multi-objective decision robustness

Recall: $\bar{x}$ is efficient if there is no $x \in \Omega$ with

$$f(x) \in \{f(\bar{x})\} - K \setminus \{0\}.$$

We replace $f(x)$ by $f_Z(x) = \{f(x+z) \in \mathbb{R}^m \mid z \in Z\}$:

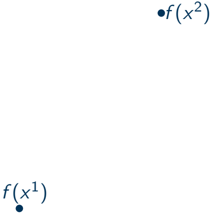
Definition

An element $\bar{x} \in \Omega$ is called *decision robust efficient* with respect to Z, if $\bar{x} \in X$ and if there is no $x \in X$ with

$$f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}.$$

Multi-objective decision robustness

Here: $m = 2$ and $K = \mathbb{R}^2_+$. Let $x^1, x^2 \in X$:



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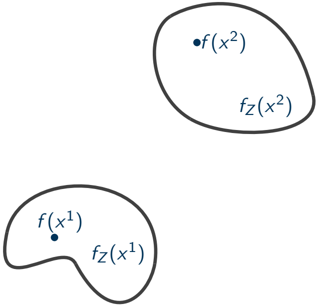

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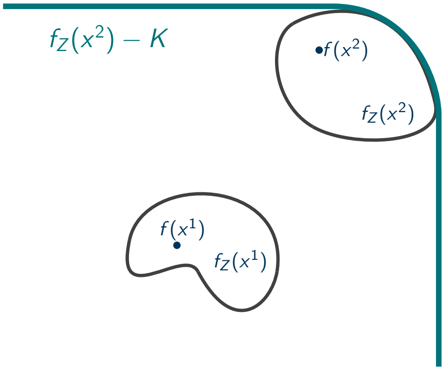

x^2 is not decision robust efficient with respect to Z as $f_Z(x^1) \subseteq f_Z(x^2) - K \setminus \{0\}$

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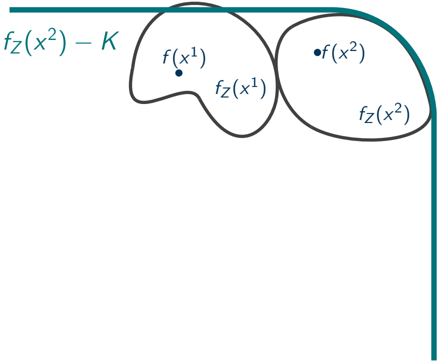

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Weak multi-objective decision robustness

Recall: $\bar{x}$ is weakly efficient if there is no $x \in \Omega$ with

$$f(x) \in \{f(\bar{x})\} - \mathrm{int}(K).$$

We replace again $f(x)$ by $f_Z(x) = \{f(x+z) \in \mathbb{R}^m \mid z \in Z\}$:

Definition

An element $\bar{x} \in \Omega$ is called *decision robust weakly efficient* with respect to Z, if $\bar{x} \in X$ and if there is no $x \in X$ with

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A simple example

Example:

Let $m = 2$ *and* $f = id$, $\Omega = B(0,3)$ *and* $Z = B(0,1)$.

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and thus $\{x + z \in \mathbb{R}^2 \mid z \in Z\} \subseteq \Omega = B(0,3) \ \Leftrightarrow \ x \in B(0,2) =: X$.

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Then all

$$\bar{x} \in \{x \in \partial X \mid x_1 \leq 0,\ x_2 \leq 0\}$$

are decision robust efficient w.r.t. Z.

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are decision robust efficient w.r.t. Z.

Note: the set of efficient solutions of $\min_{x \in X} f(x)$ *is* $\{x \in \partial X \mid x_1 \leq 0,\; x_2 \leq 0\}$.

Relation to the scalar-valued case

Lemma

For $m = 1,\ K = \mathbb{R}_+$ *we have*

$$\bar{x} \in \min_{x \in X} \sup_{z \in Z} f(x+z) \ \Leftrightarrow\ \bar{x} \text{ decision robust (weakly) efficient.}$$

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Other Robustness (minmax) approaches

- Avikad, Branke 2008, Ehrgott, Ide, Schöbel 2014: Minmax robustness for multi-objective optimization problems, **compare sets** $\{f(x,z) \mid z \in Z\}$

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- Kuroiwa, Lee, 2012, Fliege, Werner 2013; Doolittle, Kerivin, and Wiecek 2012: replace **each objective function** by the function consisting of the worst cases: $\min_{x \in X}(\max_z f_1(x+z), \ldots, \max_z f_m(x+z))$

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- Deb, Gupta 2006: minimize **mean value** of function values of δ-neighborhood of x

Decision Robust Efficiency

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- *(Def.:) there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.*
- *$\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem*

$$\min_{x \in X} f_Z(x).$$

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Thus we are doing set optimization here!

Section 3
Relation to Set Optimization

Set Optimization Problem

Y real linear space, e.g. $Y = \mathbb{R}^m$, partially ordered by
$K \subseteq Y$ a pointed convex cone, e.g. $K = \mathbb{R}^m_+$,
Ω feasible set,
$F\colon \Omega \rightrightarrows Y$ set-valued map.

A set optimization problem is an optimization problem of the form

$$\min_{x \in \Omega} F(x). \tag{SOP}$$

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Set Optimization - Vector Approach

$$\min_{x\in\Omega} F(x) \qquad \text{(SOP)}$$

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Optimality using the vector approach:

Definition

A pair $(\bar{x}, \bar{y})$ with $\bar{x} \in \Omega$ and $\bar{y} \in F(\bar{x})$ is called a minimizer of (SOP) if

$$(\{\bar{y}\} - K) \cap F(\Omega) = \{\bar{y}\},$$

where

$$F(\Omega) = \bigcup_{x \in \Omega} F(x),$$

i.e. if $\bar{y}$ is a minimal element of $F(\Omega)$.

Set Optimization - Vector Approach

Example:

Let $\Omega = \{\frac{1}{2}, 1, 2\}$, $K = \mathbb{R}^2_+$, *and*

$$
\begin{aligned}
F(2) &= \{(-t, 2t) \in \mathbb{R}^2 \mid t \in [0,2]\}. \\
F(1) &= \{(-t, t) \in \mathbb{R}^2 \mid t \in [0,2]\}, \\
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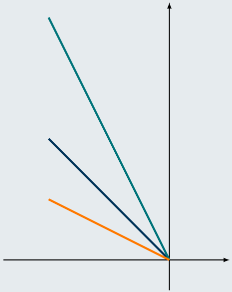
$$



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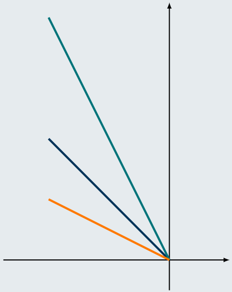

Then for instance all pairs $(\bar{x}, \bar{y})$ *with* $\bar{y} = (0,0)$ *and* $\bar{x} \in \Omega$ *are minimizers of* $\min_{x \in \Omega} F(x)$.

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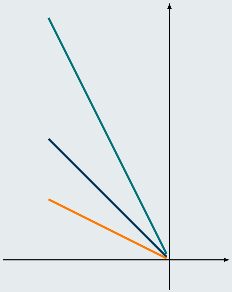

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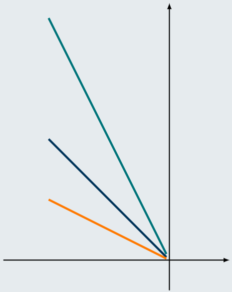

Then only pairs $(\bar{x}, \bar{y})$ *with* $\bar{y} \in F(\frac{1}{2})$ *and* $\bar{x} = \frac{1}{2}$ *are minimizers of* $\min_{x \in \Omega} F(x)$.

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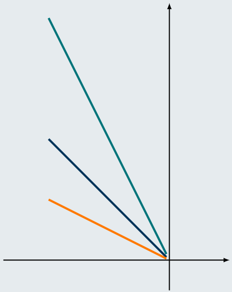

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In general only one element $\bar{y}$ does not imply that the whole set $F(\bar{x})$ is in a certain sense minimal with respect to all sets $F(x)$ with $x \in \Omega$!

Set Optimization - Set Approach

We need order relations for comparing sets! Let $A, B \subseteq \mathbb{R}^m$.

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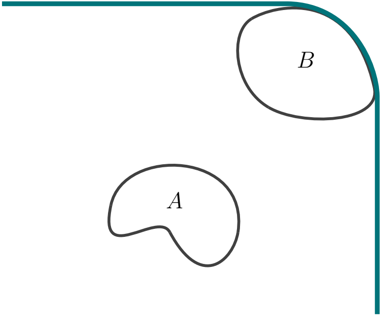
- u-type less order relation $\preccurlyeq_u$ is defined by

$$A \preccurlyeq_u B \;:\Leftrightarrow\; A \subseteq B - K.$$



Relation to Set Optimization: Set Relations

- $$\begin{aligned} A \preccurlyeq_u B \quad &:\Leftrightarrow \quad A \subseteq B - K. \\ &\Leftrightarrow \quad (\forall a \in A\ \exists b \in B : a \leqq b) \end{aligned}$$

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- $$\begin{aligned} A \preccurlyeq_{\mathrm{int}(K)} B \quad &:\Leftrightarrow \quad A \subseteq B - \mathrm{int}(\mathrm{K}). \\ &\Leftrightarrow \quad (\forall a \in A\ \exists b \in B : a < b) \end{aligned}$$

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Using these set relations we can replace for instance $f_Z(x) \subseteq f_Z(\bar{x}) - K$ in the definition of multi-objective decision robust efficiency by $f_Z(x) \preccurlyeq_u f_Z(\bar{x})$.

More Set Relations

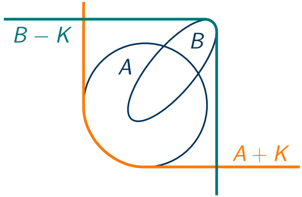
- u-type less order relation $\preccurlyeq_u$

$$\begin{aligned} A \preccurlyeq_u B \quad &:\Leftrightarrow \quad A \subseteq B - K. \\ &\Leftrightarrow \quad (\forall a \in A \ \exists b \in B : a \leqq b) \end{aligned}$$

- l-type less order relation $\preccurlyeq_l$

$$\begin{aligned} A \preccurlyeq_l B \quad &:\Leftrightarrow \quad B \subseteq A + K. \\ &\Leftrightarrow \quad (\forall b \in B \ \exists a \in A : a \leqq b) \end{aligned}$$

- set less order relation: $A \preccurlyeq_s B \ :\Leftrightarrow \ A \preccurlyeq_u B \ \wedge \ A \preccurlyeq_l B.$



Optimal Solution in Set Optimization

Definition (Rodríguez-Marín, Sama '07)

Let a nonempty set $\Omega \subseteq \mathbb{R}^n$ and a set-valued map $F\colon X \rightrightarrows \mathbb{R}^m$ be given with $F(x) \neq \emptyset$ for all $x \in \Omega$.
An element $\bar{x} \in \Omega$ is called a *strictly optimal solution* for the set-valued optimization problem

$$\min_{x \in \Omega} F(x)$$

w.r.t. $\preccurlyeq$ if

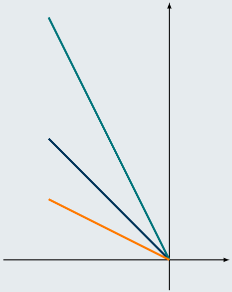
$$\text{there exists no } \; x \in \Omega \backslash \{\bar{x}\} \text{ with } \; F(x) \preccurlyeq F(\bar{x}).$$

Example: Set Approach

Example:

Let $\Omega = \{\frac{1}{2}, 1, 2\}$, $K = \mathbb{R}^2_+$, *and*

$$\begin{aligned} F(2) &= \{(-t, 2t) \in \mathbb{R}^2 \mid t \in [0,2]\}. \\ F(1) &= \{(-t, t) \in \mathbb{R}^2 \mid t \in [0,2]\}, \\ F(\tfrac{1}{2}) &= \{(-t, \tfrac{1}{2}t) \in \mathbb{R}^2 \mid t \in [0,2]\}, \end{aligned}$$



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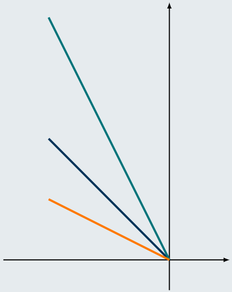

The only strictly optimal solution is $\bar{x} = \frac{1}{2}$ *(for* $\preccurlyeq_u$, $\preccurlyeq_l$, $\preccurlyeq_s$*).*

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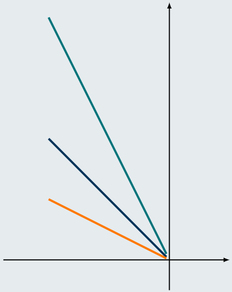

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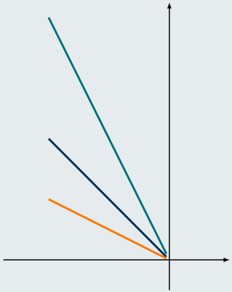

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Decision Robust Efficiency as Set Optimization p.

$$\min_{x \in X} f_Z(x) \quad \text{with} \quad f_Z \colon X \rightrightarrows \mathbb{R}^m,\ x \mapsto \{f(x+z) \in \mathbb{R}^m \mid z \in Z\}. \qquad \text{(SOP)}$$

Corollary

The point $\bar{x} \in X$ is decision robust [weakly/·] efficient w.r.t. Z if and only if $\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem (SOP) w.r.t. $[\preccurlyeq_{\mathrm{int}(K)} / \preccurlyeq_{K\setminus\{0\}}]$.

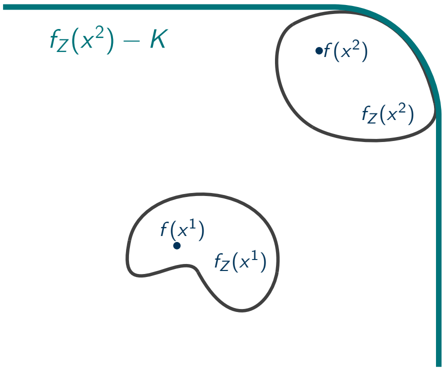
Corollary

An element $\bar{x} \in X$ is decision robust efficient with respect to Z if and only if there is no $x \in X\setminus\{\bar{x}\}$ with

$$\forall z \in Z\ \exists z' \in Z\ :\ f(x+z) \leq f(\bar{x}+z').$$

Multi-objective Decision Robust Efficiency

Here: $m = 2$ and $K = \mathbb{R}^2_+$. Let $x^1, x^2 \in X$:



x^2 is not decision robust efficient with respect to Z as
$f_Z(x^1) \preccurlyeq_{K\setminus\{0\}} f_Z(x^2)$

Decision Robust Efficiency

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- *(Def.:) there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.*
- *$\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem*
$$\min_{x \in X} f_Z(x).$$
- *$\bar{x} \in X$ is a strictly optimal solution for the set-valued optimization problem*
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Decision Robust Efficiency

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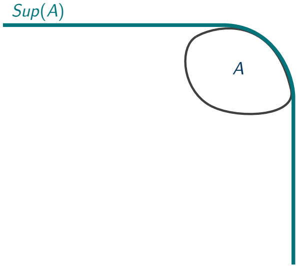
Supremal Sets

Definition (Nieuwenhuis 1980, Löhne 2011)

Let K be a pointed closed cone with $\text{int}(K) \neq \emptyset$. For a bounded nonempty subset $A \subsetneq \mathbb{R}^m$ the supremal set of A is defined as

$$\text{Sup}(A) = \{y \in \text{cl}(A - K) \mid (\{y\} + \text{int}\, K) \cap \text{cl}(A - K) = \emptyset\}.$$



If $A \subsetneq \mathbb{R}^m$ is nonempty and bounded, then

$$\text{Sup}(A) \cap A = \text{wMax}(A)$$

where $\text{wMax}(A) = \{y \in A \mid (\{y\} + \text{int}(K)) \cap A = \emptyset\}$.

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We study the set-valued optimization problem

$$\min_{x \in X} f_{Sup}(x) \quad \text{with} \quad f_{Sup} \colon X \rightrightarrows \mathbb{R}^m,\ x \mapsto \operatorname{Sup}(f_Z(x)).$$

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By our assumptions we have

$$f_Z(x) - K = \operatorname{cl}(f_Z(x) - K) \notin \{\mathbb{R}^m, \emptyset\}.$$

which implies (see Löhne 2011) that $f_{Sup}(x) \neq \emptyset$ for all $x \in X$.

Set Relations and Supremal Sets

Lemma

Let $A, B \subsetneq \mathbb{R}^m$ be compact sets. Then

$$A - K = \mathrm{Sup}\,(A) - K \quad \text{and} \quad A - K \setminus \{0\} = \mathrm{Sup}\,(A) - K \setminus \{0\}$$

and for any $\preccurlyeq \in \{\preccurlyeq_u, \preccurlyeq_{K\setminus\{0\}}, \preccurlyeq_{\mathrm{int}(K)}\}$ *it holds*

$$A \preccurlyeq B \qquad \Leftrightarrow \qquad \mathrm{Sup}\,(A) \preccurlyeq \mathrm{Sup}\,(B).$$

Decision Robustness by 'min sup'

Theorem

The point $\bar{x} \in X$ is decision robust efficient if and only if

- *there is no $x \in X$ with $f_Z(x) \subseteq f_Z(\bar{x}) - K \setminus \{0\}$.*

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w.r.t. $\preccurlyeq_{K\setminus\{0\}}$.

Section 4
(Numerical) Approaches to Set Optimization

Stronger Concepts for Necessary Conditions

- certainly less order relation $\preccurlyeq_c$

$$A \preccurlyeq_c B \quad :\Leftrightarrow \quad B \subseteq \bigcap_{a\in A}\{a\} + \mathbb{R}^m_+.$$

$$\Leftrightarrow \quad (A = B) \text{ or } (A \neq B,\ \forall a \in A,\ \forall b \in B:\ \ a \leqq b)$$

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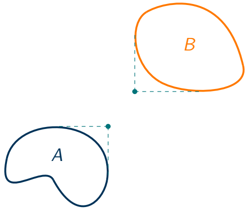

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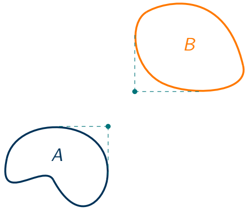
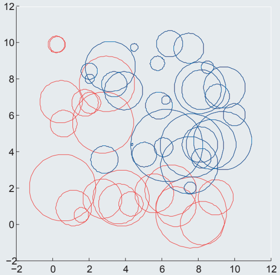


It holds: $A \preccurlyeq_c B \ \Rightarrow \ A \preccurlyeq_s B.$

Necessary Conditions

Example:

Let $\mathcal{F}$ be a family of n disks with random radius $r \in]0,2[$ and random center points $(x_i, y_i) \in]0,10[\times]0,10]$, $i = 1, \ldots, n$.



n	50	100	500	1000	5000
k	30	66	371	755	3941
$\frac{k}{n}$	0.61	0.66	0.74	0.76	0.79

where k is the number of disks that are larger than another disk w.r.t. $\preccurlyeq_c$.

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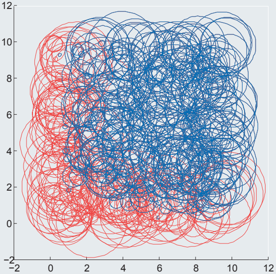

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Elements from the Dual Cone

Sets $\emptyset \neq A, B \subseteq \mathbb{R}^m$ compact and convex. Then

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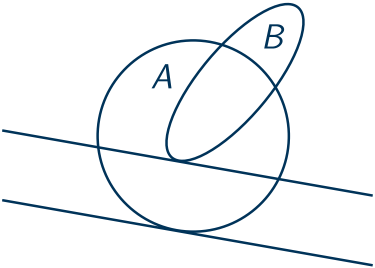

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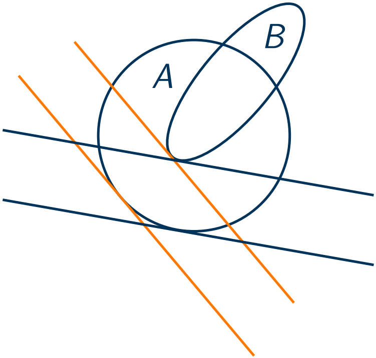

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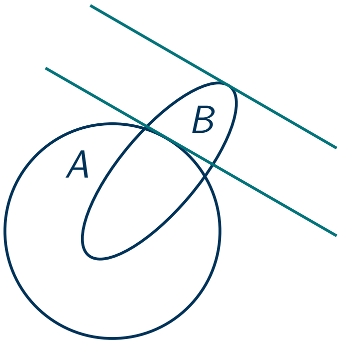

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Vectorization

Define for compact and convex sets A

$$v(A)(w) := \begin{pmatrix} \min\limits_{a \in A} w^\top a \\ \max\limits_{a \in A} w^\top a \end{pmatrix}.$$

Then

$$A \preccurlyeq_s B \;\Leftrightarrow\; v(A)(w) \leq v(B)(w) \;\forall\; w \in \mathbb{R}^m_+ \text{ with } \|w\| = 1.$$

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- derivative free descent method
- better results for $F(x)$ polyhedral:
  $F(x) := \{y \in \mathbb{R}^m \mid A(x) \cdot y \le b(x)\}$ and
  $A(x) \in \mathbb{R}^{p\times m},\ b(x) \in \mathbb{R}^p$ for all $x \in \Omega$

Nonlinear Scalarization

[Gutiérrez, Jiménez, Miglierina, Molho '15]

Choose $r \in \mathbb{R}^m_+ \setminus \{0\}$ and define to sets $\emptyset \neq A, B \subseteq \mathbb{R}^m$:

$$\begin{aligned} g_B(A) &:= \inf\{t \in \mathbb{R} \mid t\, r + B \subseteq A + \mathbb{R}^m_+\} \\ &= \sup_{b \in B} \inf_{a \in A} (\inf\{t \in \mathbb{R} \mid t\, r + b \in a + K\}). \end{aligned}$$

Then $\bar{x}$ is a strictly optimal solution for the set-valued optimization problem $\min_{x \in \Omega} F(x)$ if

$$g_{F(\bar{x})}(F(x)) > 0 \text{ for all } x \in \Omega \setminus \{\bar{x}\}.$$

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