

Finding bicriteria shortest pairs of disjoint paths: algorithms and related problems¹

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Contents

Abstract

Motivation

The shortest pair of disjoint paths problem

Ranking shortest pairs of disjoint paths

The shortest pair of arc/label disjoint paths problem

The bicriteria shortest pair of arc/label disjoint paths problem

Approach 1: 2-phases method

Approach 2: ranking based method

Computational experiments

Abstract

Pairs of disjoint paths between two nodes in a network provide a primary path as well as a backup path, such that the first can be used in case one of the arcs of the second fails. In this talk we address the bicriteria shortest pair of disjoint paths problem and some related problems, and discuss exact algorithms to find their non-dominated solutions.

Motivation

1

- ▶ Sometimes it is convenient to know an alternative to the shortest path, in case a failure occurs along the chosen primary path. They should not share physical resources.
- ▶ In telecommunications, whenever it justifies to have a backup path.
- ▶ In transportation, in order to avoid over-exposure of the same region to hazardous materials.

2

- ▶ Additionally, some applications associate each arc with a type of service, which require types not to be repeated in the main and the backup paths, in order to prevent possible failures.

Something old

The shortest pair of disjoint paths problem

Definitions

- ▶ $G = (N, A)$: **network**, $A \subseteq N \times N$, $|N| = n$, $|A| = m$.
- ▶ $c_{ij} \in \mathbb{R}$: **cost** associated with $(i, j) \in A$.
- ▶ $s \in N$ ($t \in N$): **source (terminal)** node.
- ▶ Given a path p , the set of nodes (arcs) in p is $V(p)$ ($A(p)$).
- ▶ $p \diamond q$ denotes the **concatenation** of paths p and q .

Assumptions

- ▶ No parallel arcs.
- ▶ No cycles with a negative cost.

The shortest pair of disjoint paths problem

Let $s, t \in N$, the set of pairs of disjoint paths from s to t is

$$L = \left\{ x \in \{0, 1\}^m : \sum_{(i,j) \in \Gamma^+(i)} x_{ij} - \sum_{(j,i) \in \Gamma^-(i)} x_{ji} = b_i, \forall i \in N, \forall (i,j) \in A \right\}$$

with

$$b_i = \begin{cases} 2 & \text{if } i = s \\ 0 & \text{if } i \in N - \{s, t\} \\ -2 & \text{if } i = t \end{cases}$$

and $x_{ij} = 1$ iff $(i,j) \in A$ belongs to a pair of disjoint paths.

The constraints matrix is totally unimodular and b is integral, thus L is

$$\left\{ x \in \mathbb{R}^m : \sum_{(i,j) \in \Gamma^+(i)} x_{ij} - \sum_{(j,i) \in \Gamma^-(i)} x_{ji} = b_i, \forall i \in N \wedge 0 \leq x_{ij} \leq 1, \forall (i,j) \in A \right\}$$

The shortest pair of disjoint paths problem

The problem of finding a shortest pair of disjoint paths from s to t (SPDP) can be formulated as:

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j) \in A} c_{ij} x_{ij} \\ \text{subject to} & x \in L \end{array}$$

This is a **minimum cost flow problem** in a network, where:

- ▶ the capacity upper bound of (i, j) is $u_{ij} = 1$,
- ▶ $b_s = 2$, $b_t = -2$, $b_i = 0$, $\forall i \in N - \{s, t\}$.

Finding the shortest pair of disjoint paths

Suurballe's algorithm

Thus, the SPDP can be found by considering $x = 0$, and

1. finding a shortest path in G , and
2. finding a shortest path in a residual network $G[x] = (N, A_x, c^x)$, s.t. $A_x = A_+ \cup A_-$, and

$$\blacktriangleright A_+ = \overbrace{\{(i, j) : (i, j) \in A \wedge x_{ij} = 0\}}^{c_{ij}^x = c_{ij}}$$

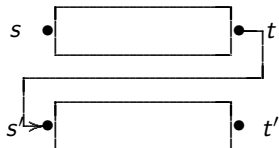
$$\blacktriangleright A_- = \overbrace{\{(j, i) : (i, j) \in A \wedge x_{ij} = 1\}}^{c_{ji}^x = -c_{ij}}$$

2 shortest path problems are solved $\Rightarrow O(m + n \log n)$ time

Ranking shortest pair of disjoint paths

New network (N', A') [Clímaco & Pascoal, 2009]:

- ▶ **Nodes** are duplicated: $N' = N \cup \{i' : i \in N\}$.
- ▶ **Arcs** are duplicated and a new one is added, from t to s' :
 $A' = A \cup \{(i', j') : (i, j) \in A\} \cup \{(t, s')\}$.
- ▶ **Initial node**: s , **Terminal node**: t' .
- ▶ Old **costs** are maintained. New costs: $c_{i'j'} = c_{ij}$, if $(i, j) \in A$, and $c_{ts'} = 0$.



Ranking shortest pair of disjoint paths

Paths in (N', A')

- ▶ Each path p from s to t' has the form:

$$p = q \diamond (t, s') \diamond q',$$

where q is a path from s to t and q' is a path from s' to t' .

- ▶ If q and q' are simple and $q \cap q' = \emptyset$, then q, q' correspond to a pair of disjoint paths in (N, A) (maybe simple).

Algorithm

A method for ranking simple paths can be applied to the modified topology, in order to compute disjoint pairs of paths by order of cost.

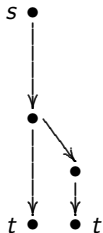
Ranking shortest pair of disjoint paths

Deviation algorithm [Martins et al., 1999]

Let X be a set of candidates to the next p_k , $k \in \{1, \dots, K\}$.

- ▶ X is initialised with p_1 .
- ▶ Repeat:
 - ▶ p_k is the shortest simple path in X ,
 - ▶ analyse p_k and generate new low cost candidates from it,
 - ▶ store the candidates generated in X .

New candidates are deviations.



Ranking shortest pair of disjoint paths

The shortest path from s to t that deviates from p_k at v_i is:

$$\text{sub}_{p_k}(s, v_i) \diamond (v_i, j) \diamond \mathcal{T}_t(j),$$

- s. t. (v_i, j) doesn't belong to any of the candidates computed. Besides:
- ▶ $\mathcal{T}_t(j)$ can be computed when the algorithm starts,
 - ▶ the access to (v_i, j) can be immediate, and
 - ▶ $j \notin \text{sub}_{p_k}(s, v_i)$, $h(j) \notin \text{sub}_{p_k}(s, v_i) \cap \mathcal{N}$ if $j \in \mathcal{N}'$ where $h(x') = x$ for any $x \in \mathcal{N}$.

Check if deviations correspond to disjoint pairs of paths!

Something new

The shortest pair of arc/label disjoint paths problem

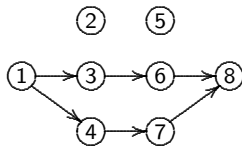
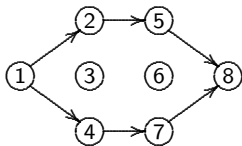
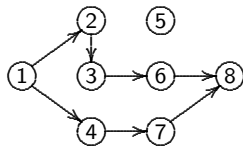
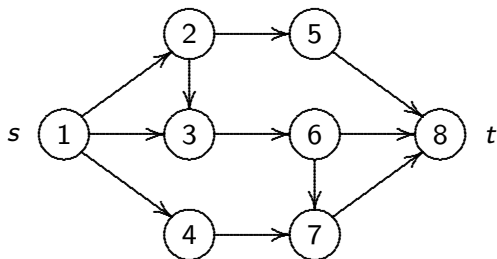
Definitions

- ▶ $c_{ij} \in \mathbb{R}$: **cost** associated with $(i, j) \in A$;
- ▶ $l_{ij} \in \mathcal{L} = \{1, \dots, \ell\}$: **label** associated with $(i, j) \in A$.

Decision variables

- ▶ $x_{ij}^k = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the path } k \\ 0 & \text{otherwise} \end{cases}, \quad \forall (i, j) \in A$
- ▶ $v_l^k = \begin{cases} 1 & \text{if label } l \text{ is in some arc of the path } k \\ 0 & \text{otherwise} \end{cases}, \quad \forall l \in \mathcal{L}$

The shortest pair of arc/label disjoint paths problem



The shortest pair of arc/label disjoint paths problem

$$\begin{aligned}
 \min \quad & \sum_{(i,j) \in \mathcal{A}} c_{ij} \sum_{k=1}^2 x_{ij}^k \\
 \text{s. t.} \quad & \sum_{j \in \Gamma^+(i)} x_{ij}^k - \sum_{j \in \Gamma^-(i)} x_{ji}^k = \begin{cases} 1, & i = s \\ 0, & i \in \mathcal{N} - \{s, t\} \\ -1, & i = t \end{cases}, & k = 1, 2 \\
 & \sum_{k=1}^2 x_{ij}^k \leq 1, & (i,j) \in \mathcal{A} \\
 & \sum_{(i,j) \in \mathcal{A}_l} x_{ij}^k \leq \min\{n-1, |\mathcal{A}_l|\} v_l^k, & l \in \mathcal{L}, k = 1, 2 \\
 & \sum_{k=1}^2 v_l^k \leq 1, & l \in \mathcal{L} \\
 & x_{ij}^k \in \{0, 1\}, & (i,j) \in \mathcal{A}, k = 1, 2 \\
 & v_l^k \in \{0, 1\}, & l \in \mathcal{L}, k = 1, 2
 \end{aligned}$$

The shortest pair of arc/label disjoint paths problem

$$1. \quad \sum_{j \in \Gamma^+(i)} x_{ij}^k - \sum_{j \in \Gamma^-(i)} x_{ji}^k = \begin{cases} 1, & i = s \\ 0, & i \in \mathcal{N} - \{s, t\} \\ -1, & i = t \end{cases}, \quad k = 1, 2$$

ensure flow conservation for all nodes and that paths start at s and end at t .

$$2. \quad \sum_{k=1}^2 x_{ij}^k \leq 1, \quad (i, j) \in \mathcal{A} \text{ assure disjointness.}$$

$$3. \quad \sum_{(i,j) \in \mathcal{A}_l} x_{ij}^k \leq \min\{n-1, |\mathcal{A}_l|\} v_l^k, \quad l \in \mathcal{L}, \quad k = 1, 2$$

assure that for each label and each path, an arc with that label can only be in the solution if the associated label variable is 1.

It also implies that the number of arcs in each path in the solution with each label cannot exceed $n-1$, neither the number of arcs with that label in the network.

The shortest pair of arc/label disjoint paths problem

$$4. \sum_{k=1}^2 v_l^k \leq 1, l \in \mathcal{L}$$

assure that each labels appears in at most one of the two paths.

$$5. x_{ij}^k \in \{0, 1\}, (i, j) \in \mathcal{A}, k = 1, 2$$
$$v_l^k \in \{0, 1\}, l \in \mathcal{L}, k = 1, 2$$

are the integrality constraints.

The shortest pair of arc/label disjoint paths problem

Remarks

1. Label disjoint paths \Rightarrow arc disjoint paths. Therefore, we can omit

$$\sum_{k=1}^2 x_{ij}^k \leq 1, \quad (i,j) \in \mathcal{A}$$

2. It can be shown that there is always an integral optimal solution x , even if

$$x_{ij}^k \in \{0, 1\}, \quad (i,j) \in \mathcal{A}, \quad k = 1, 2$$

are omitted. Therefore, these can be replaced by

$$0 \leq x_{ij}^k \leq 1, \quad (i,j) \in \mathcal{A}, \quad k = 1, 2$$

The shortest pair of arc/label disjoint paths problem

Comparison (CPU times in sec.s)

Computer: Dual Core AMD, 2.7 GHz, 4 Gb of RAM, Suse 10.3.

n	d	ℓ	F	FL	S	SL
100	5	5	0.070	0.057	0.070	0.043
100	5	10	0.057	0.038	0.052	0.030
100	5	20	0.034	0.023	0.035	0.021
100	10	5	0.129	0.099	0.096	0.068
100	10	10	0.087	0.057	0.063	0.046
100	10	20	0.052	0.027	0.083	0.047
100	20	5	0.213	0.187	0.138	0.123
100	20	10	0.135	0.097	0.098	0.067
100	20	20	0.116	0.101	0.102	0.086
200	5	5	0.116	0.066	0.090	0.047
200	5	10	0.055	0.051	0.043	0.035
200	5	20	0.052	0.035	0.053	0.029
200	10	5	0.206	0.146	0.136	0.103
200	10	10	0.205	0.180	0.142	0.118
200	10	20	0.168	0.134	0.136	0.117

n	d	ℓ	F	FL	S	SL
200	20	5	0.586	0.490	0.282	0.250
200	20	10	0.386	0.457	0.275	0.229
200	20	20	0.300	0.217	0.226	0.170
500	5	5	0.350	0.163	0.245	0.142
500	5	10	0.312	0.228	0.212	0.188
500	5	20	0.210	0.119	0.174	0.114
500	10	5	0.994	0.647	0.326	0.264
500	10	10	0.567	0.531	0.447	0.320
500	10	20	0.465	0.502	0.363	0.344
500	20	5	0.952	0.800	0.523	0.365
500	20	10	1.020	0.918	0.600	0.566
500	20	20	0.659	0.530	0.503	0.346

F: original model

FL: original model + relaxed x

S: shortened model

SL: shortened model + relaxed x

The bicriteria shortest pair of arc/label disjoint paths problem

Definitions

Under the previous conditions, but replacing c_{ij} by

- ▶ $c_{ij}^1, c_{ij}^2 \in \mathbb{R}$: **costs** associated with $(i, j) \in A$.

The bicriteria shortest pair of arc/label disjoint paths problem (BSPLDP) consists in:

$$\begin{aligned} \text{minimize} \quad & c^1(x) = \sum_{(i,j) \in A} c_{ij}^1 \sum_{k=1}^2 x_{ij}^k, \quad c^2(x) = \sum_{(i,j) \in A} c_{ij}^2 \sum_{k=1}^2 x_{ij}^k \\ \text{subject to} \quad & \text{the previous constraints} \end{aligned}$$

The bicriteria shortest pair of arc/label disjoint paths problem

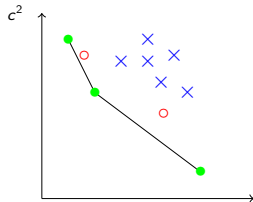
- ▶ A feasible solution x is **efficient** iff there is no other x' s.t. $c(x') \leq c(x)$ and $c(x') \neq c(x)$ (with $c(x) = (c^1(x), c^2(x))$).
- ▶ $c(x)$ is **non-dominated** iff x is efficient.
- ▶ The **Pareto frontier** is formed by all efficient solutions.

Non-dominated solutions:

- ▶ **supported**: optimal solutions of a weighted sum problem (WSP)

$$\min_{x \in S} \{w_1 c^1(x) + w_2 c^2(x) : w_1, w_2 \geq 0\}$$

- ▶ the remaining are **unsupported**.



- × dominated solution c^1
- unsupported solution
- supported solution

Something “borrowed”

Approach 1: 2-phases method

Two phases:

1. compute the supported solutions,
2. compute the unsupported solutions, by looking for solutions with images within the duality gaps formed by adjacent supported solutions.

Approaches for searching within duality gaps, [Current et al., 1990], [Ralphs, Saltzman & Wiecek, 2006]:

- ▶ using **side constraints** to obtain a sequence of “easier” problems,
- ▶ using an **algorithm that ranks the K best solutions** of the problem,
- ▶ using a reference point approach, minimizing a Chebyshev metric,
- ...

Approach 1: 2-phases method

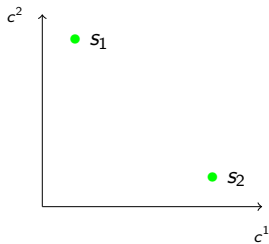
The WSM approach solves a sequence of WSPs and updates their parameters according to the solutions found (NISE method [Cohon, 1978]).

1. The lexicographic best solution wrt each criterion is calculated.
2. For each pair of consecutive solutions, s_1 , s_2 , a new one is calculated. The goal is to find further non-dominated solutions inside the gap defined by $c(s_1)$ and $c(s_2)$. The new solution optimizes

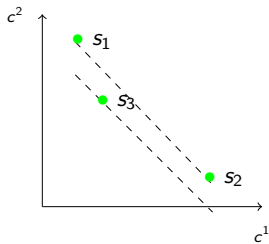
$$\begin{array}{ll} \min & w_1 c^1(s) + w_2 c^2(s) \\ \text{s. t.} & s \in S \end{array}$$

with $w_1 = c^2(s_1) - c^2(s_2)$, $w_2 = c^1(s_2) - c^1(s_1)$.

Approach 1: 2-phases method



(a)



(b)

Phase 1, WSM for computing supported solutions:

- (a) initial supported solutions;
- (b) supported solution obtained from the previous pair of solutions.

Approach 1: 2-phases method

- ▶ Other solutions belong to the duality gaps formed by pairs of adjacent supported solutions.
- ▶ These unsupported solutions can be obtained by solving a new sequence of problems and updating the parameters defined by the search regions according to their solutions.

Given s_1, s_2 adjacent non-dominated solutions, the method solves

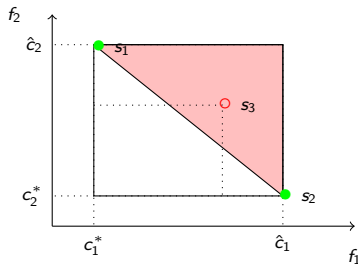
$$\begin{array}{ll} \min & \max\{w_1|c^1(s) - c_1^*|, w_2|c^2(s) - c_2^*|\} \\ \text{s. t.} & s \in S \end{array}$$

with $c_i^* = c^i(s_i)$, $i = 1, 2$, which depends on the search region.

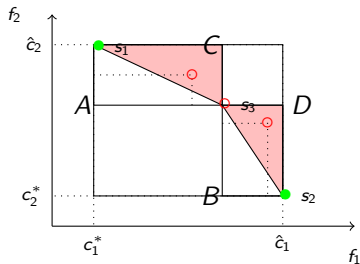
Equivalently, we have the LP

$$\begin{array}{ll} \min & v \\ \text{s. t.} & w_1 f_2(s) - v \leq w_1 f_2(s_2) \\ & w_2 f_1(s) - v \leq w_2 f_1(s_1) \\ & s \in S \end{array}$$

Approach 1: 2-phases method



(a)



(b)

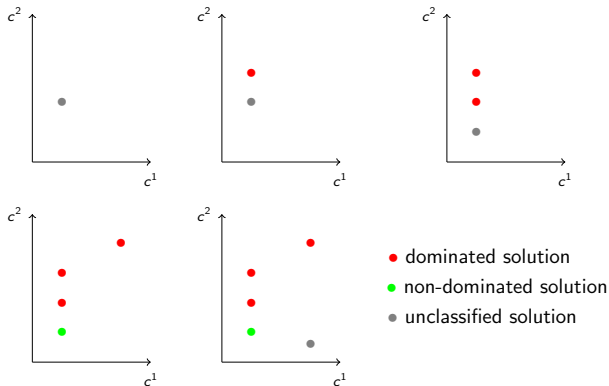
Phase 2, computing unsupported solutions:

- (a) initial search region;
- (b) regions to scan after the unsupported solution s_3 is calculated in search for possible non-dominated solutions.

Approach 2: ranking based method

Initially for the bicriteria path problem [Clímaco & Martins, 1982]:

- ▶ Solutions are ranked by order of c^1 (or c^2).
- ▶ A test filters the dominated solutions.



Approach 2: ranking based method

Procedure

- ▶ $\hat{c}_1 \leftarrow$ best c_1 value of a solution
- ▶ $c_2^* \leftarrow$ worst c_2 value of a solution
- ▶ $m_2 \leftarrow \hat{c}_2 \leftarrow$ best c_2 value of a solution
- ▶ $M_1 \leftarrow c_1^* \leftarrow$ worst c_1 value of a solution
- ▶ To rank solutions by order of c_1 and filter non-dominated solutions by means of a dominance test

This method can be extended to other problems, provided a ranking algorithm is known.

Approach 2: ranking based method

Ranking algorithm

The algorithm by [Clímaco & Pascoal, 2009] can be adapted.

In order to ensure that the pairs of disjoint paths are label disjoint:

- ▶ The selection of the deviation arc is restricted, so that the labels in the 2 paths are different.
- ▶ Candidates with repeated labels are discarded, after use.

Machine and instances

Code in C language, running over Suse 10.3.

Computer characteristics:

- ▶ Dual Core AMD
- ▶ Processor 2.7 GHz
- ▶ 4 Gbytes of RAM

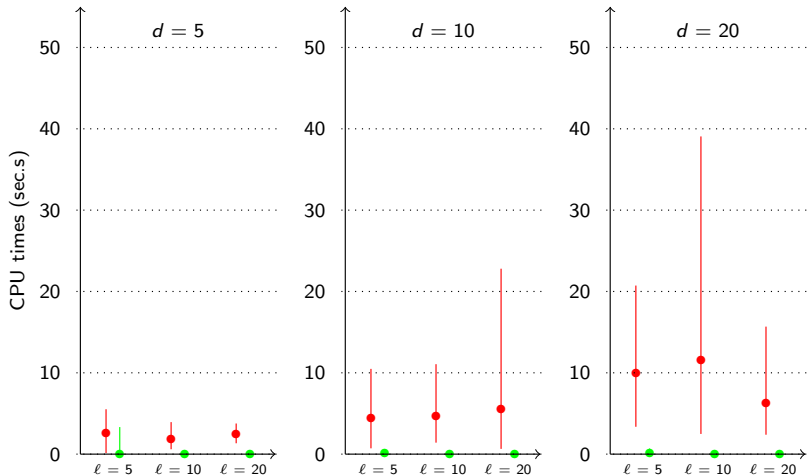
Random networks (generated with 10 seeds):

- ▶ $n \in \{100, 200, 50\}$
- ▶ $d \in \{5, 10, 20\}$
- ▶ c_{ij}^1, c_{ij}^2 randomly generated in $\{0, \dots, 100\}$

CPU times: $n = 100$

— Approach 1

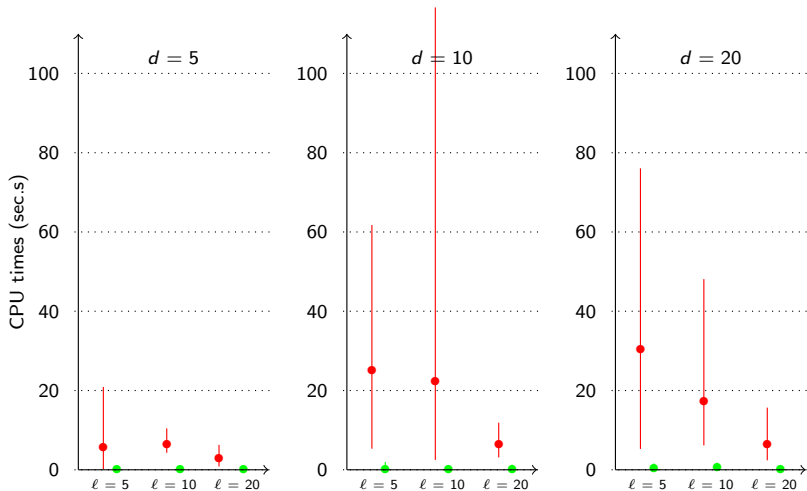
— Approach 2



CPU times: $n = 200$

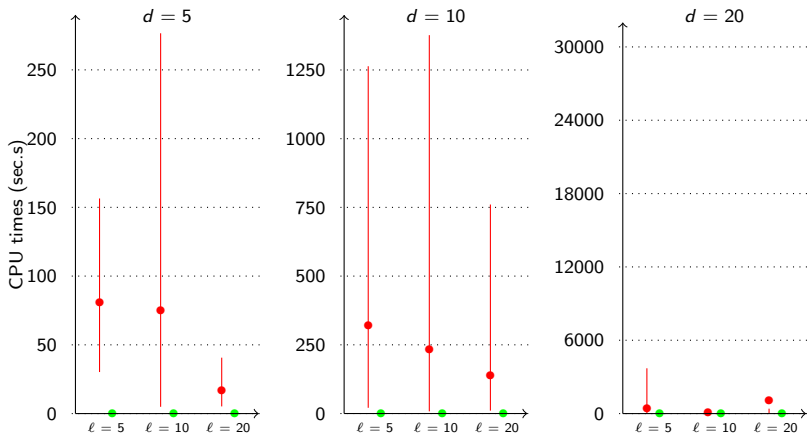
— Approach 1

— Approach 2



CPU times: $n = 500$

— Approach 1 — Approach 2



Number of non-dominated solutions

			# n.-dom. sol.s			Succ.
<i>n</i>	<i>d</i>	<i>ℓ</i>	Mean	min	Max	(%)
100	5	5	4.545	2	9	20
100	5	10	4.067	2	10	55
100	5	20	7.071	3	16	60
100	10	5	8.000	3	19	65
100	10	10	10.294	5	20	90
100	10	20	11.611	5	17	80
100	20	5	13.500	6	23	55
100	20	10	14.450	8	23	70
100	20	20	18.000	10	27	80
200	5	5	3.500	2	7	50
200	5	10	6.857	2	11	60
200	5	20	5.917	2	14	65
200	10	5	10.500	3	22	90
200	10	10	11.556	4	29	95
200	10	20	12.250	3	24	100

			# n.-dom. sol.s			Succ.
<i>n</i>	<i>d</i>	<i>ℓ</i>	Mean	min	Max	(%)
200	20	5	14.550	8	25	100
200	20	10	15.650	5	24	100
200	20	20	17.450	7	32	100
500	5	5	6.545	2	10	35
500	5	10	6.071	2	12	65
500	5	20	9.667	2	16	35
500	10	5	10.400	4	21	40
500	10	10	12.000	4	22	80
500	10	20	13.700	3	25	60
500	20	5	13.200	6	26	35
500	20	10	18.600	10	26	30
500	20	20	22.750	5	35	60

Succ.: Approach 2 success rate (problems solved till the end.)

The end

Thank you for listening!

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